Probability III – 2008/09

Solutions to Exercise Sheet 8

1. (a) If $X(t) = k$ then the probability that the number of helicopters would be increased by 1 after a short time h is:

$$P\{X(t) = k + 1 \,|\, X(t) = k\} = [2kh + o(h)] \ + \ [h + o(h)] = (2k + 1)h + o(h),$$

where the first term is due to the word-of-mouth advertising (each existing customer gives 2 "births" per month) and the second term is due to media advertising (1 "birth" per month). Hence $X(t)$ can be modeled as a pure birth process with $\lambda_k = 1 + 2k$.

(b) This probability is $p_2(2)$, where $p_n(t) = P(X(t) = n|X(0) = 0)$.

It follows from the differential equations for the $p_n(t)$ (see lecture) that

$$p_0(t) = e^{-\lambda_0 t}$$

and

$$p_n(t) = \lambda_{n-1} e^{-\lambda_n t} \int_0^t e^{\lambda_n x} p_{n-1}(x)\, dx, \qquad n = 1, 2, \ldots.$$

Thus

$$p_1(t) = e^{-\lambda_1 t} \lambda_0 \int_0^t e^{\lambda_1 x} p_0(x)\, dx = e^{-3t} \int_0^t e^{3x} e^{-x}\, dx = \frac{1}{2}(e^{-t} - e^{-3t}),$$

and

$$\begin{aligned} p_2(t) = e^{-\lambda_2 t} \lambda_1 \int_0^t e^{\lambda_2 x} p_1(x)\, dx \ &= \ \frac{3}{2} e^{-5t} \int_0^t e^{5x} (e^{-x} - e^{-3x})\, dx \\ &= \ \frac{3}{8} e^{-t} - \frac{3}{4} e^{-3t} + \frac{3}{8} e^{-5t} \end{aligned}$$

and $p_2(2) = 0.0489$

2. (a) $T = \max(T_1, \ldots, T_N)$. Hence

$$\text{“}T \le t\text{”} \iff \text{“}T_k \le t \text{ for all } k = 1, \ldots, N\text{”}.$$

Therefore

$$\begin{aligned} P(T \le t) &= P(T_1 \le t, \ldots, T_N \le t) \\ &= P(T_1 \le t) \cdot \ldots \cdot P(T_N \le t) \quad \text{[by independence]} \\ &= [1 - \exp(-t/\theta)]^N \qquad \text{[since } P(T_k \le t) = 1 - \exp(-t/\theta) \text{ for all } k]. \end{aligned}$$

The c.d.f. for T is

$$F_T(t) = P(T \le t) = [1 - \exp(-t/\theta)]^N.$$

(b) Remember that if a random variable Y has an exponential distribution (say, with parameter β), then $P\{Y > t + h \,|\, Y > t\} = P\{Y > h\}$. Indeed,

$$\begin{aligned} P\{Y > t+h \,|\, Y > t\} &= \frac{P\{Y > t+h\,,\, Y > t\}}{P\{Y > t\}} \\ &= \frac{P\{Y > t+h\}}{P\{Y > t\}} = \frac{e^{-\beta(t+h)}}{e^{-\beta t}} = e^{-\beta h} = P\{Y > h\}. \end{aligned}$$

Hence for any j

$$P\{T_j > t+h \,|\, T_j > t\} = P\{T_j > h\} = e^{-\frac{h}{\theta}}.$$

In other words, if $X(t) = i$, then the behaviour of the system after time t depends on i but does not depend on t. In particular,

$$P\{X(t+h) = i \,|\, X(t) = i\} = P\{T_1 > h, ..., T_i > h\} = \prod_{j=1}^{i} P\{T_j > h\} = e^{-\frac{ih}{\theta}} = 1 - \frac{ih}{\theta} + o(h).$$

In other words, the death process $X(t)$ has parameters $\mu_i = \frac{i}{\theta}$.

(c)

$$T = S_N + S_{N-1} + \ldots + S_1,$$

where S_i, $i = N, N-1, \ldots, 1$, is the duration of time when there are exactly i individuals in the population.

Since $X(t)$ is a pure death process, S_i, $i = N, N-1, \ldots, 1$, has exponential distribution $\text{Exp}(i/\theta)$. Hence $E(S_i) = \theta/i$ and

$$\begin{aligned} E(T) &= E(S_N) + E(S_{N-1}) + \ldots + E(S_1) \\ &= \theta \left[\frac{1}{N} + \frac{1}{N-1} + \ldots + \frac{1}{1} \right]. \end{aligned}$$

3. We have $N+1$ states: 0, 1, ..., N. Then

$$\begin{aligned} P_{ij}(t+\Delta t) &= \sum_{k=0}^{N} P_{ik}(t) P_{kj}(\Delta t) \quad \text{[Chapman-Kolmogorov relation]} \\ &= P_{i,j-1}(t) P_{j-1,j}(\Delta t) + P_{ij}(t) P_{jj}(\Delta t) + P_{i,j+1}(t) P_{j+1,j}(\Delta t) + \\ &\quad \sum{}' P_{ik}(t) P_{kj}(\Delta t), \end{aligned}$$

The summation in $\sum'$ is over all $k \neq j, j \pm 1$. From Postulates defining the birth and death process

$$P_{kj}(\Delta t) = o(\Delta t) \qquad \text{for all } k \neq j, j \pm 1.$$

As usual, it follows from this relation that the finite sum

$$\sum{}' P_{ik}(t) P_{kj}(\Delta t) = o(\Delta t)$$

and

$$\begin{aligned} P_{ij}(t+\Delta t) &= P_{i,j-1}(t) P_{j-1,j}(\Delta t) + P_{ij}(t) P_{jj}(\Delta t) + P_{i,j+1}(t) P_{j+1,j}(\Delta t) + o(\Delta t) \\ &= P_{i,j-1}(t) \lambda_{j-1} \Delta t + P_{i,j}(t)[1 - (\lambda_j + \mu_j)\Delta t] + P_{i,j+1}(t) \mu_{j+1} \Delta t + o(\Delta t) \end{aligned}$$

Therefore

$$\frac{P_{ij}(t+\Delta t) - P_{ij}(t)}{\Delta t} = \lambda_{j-1}P_{i,j-1}(t) - (\lambda_j + \mu_j)P_{i,j}(t) + \mu_{j+1}P_{i,j+1}(t) + \frac{o(\Delta t)}{\Delta t}.$$

As $\Delta t \to 0$ the right hand side of this relation converges (remember the definition of $o(\Delta t)$). Hence, the left hand side converges too and this implies that the derivative $P'_{ij}(t)$ exists and, moreover,

$$P'_{ij}(t) = \lambda_{j-1}P_{i,j-1}(t) - (\lambda_j + \mu_j)P_{i,j}(t) + \mu_{j+1}P_{i,j+1}(t).$$

We thus proved that the forward Kolmogorov differential equations hold true.

Note that the above argument applies to the case when $j = 0$ if we take into account that $\mu_0 = 0$ and remember that $P_{i,-1}(t) \equiv 0$ by definition. Indeed, in this case we obtain:

$$P'_{i,0}(t) = \lambda_{-1}P_{i,-1}(t) - (\lambda_0 + \mu_0)P_{i,0}(t) + \mu_1 P_{i,1}(t) \equiv -\lambda_0 P_{i,0}(t) + \mu_1 P_{i,1}(t).$$

Similarly, if $j = N$, then $\lambda_N = 0$ and $P_{i,N+1}(t) \equiv 0$. Hence

$$P'_{i,N}(t) = \lambda_{N-1}P_{i,N-1}(t) - (\lambda_N + \mu_N)P_{i,N}(t) + \mu_{N+1}P_{i,N+1}(t) \equiv \lambda_{N-1}P_{i,N-1}(t) - \mu_N P_{i,N}(t).$$

4. (a) We know (see lecture) that the equilibrium distribution $(W_0, W_1, ..., W_N)$ satisfies the following system of equations:

$$\begin{cases} -\lambda_0 W_0 + \mu_1 W_1 \\ \lambda_{j-1}W_{j-1} - (\lambda_j + \mu_j)W_j + \mu_{j+1}W_j = 0, & \text{if } 1 \le j \le N-1 \\ \lambda_{N-1}W_{N-1} - \mu_N P_N = 0, \end{cases} \tag{1}$$

where (as in lectures) $W_j = \lim_{t\to\infty} P_{i,j}(t)$ and this limit does not depend on i. The above equations follow from the forward Kolmogorov differential equations (the proof is the same as the one explained in lectures for the case of infinitely many equations). And apart of that we should have

$$W_0 + W_1 + ... + W_N = 1. \tag{2}$$

It is easy to check via direct calculation that

$$W_j = \frac{\lambda_0}{\mu_1} \cdots \frac{\lambda_{j-1}}{\mu_j} W_0$$

solve (1) (in fact, these expressions can be derived from (1)). Taking (2) into account, one obtains

$$W_j = \frac{\frac{\lambda_0}{\mu_1} \cdots \frac{\lambda_{j-1}}{\mu_j}}{1 + \sum_{k=1}^N \frac{\lambda_0}{\mu_1} \cdots \frac{\lambda_{k-1}}{\mu_k}}.$$

(b) Substituting the birth and death parameters into formulae for W_j, we obtain:

$$W_j = \frac{\frac{\alpha N}{\beta} \cdots \frac{\alpha(N-j+1)}{\beta j}}{1 + \sum_{k=1}^N \frac{\alpha N}{\beta} \cdots \frac{\alpha(N-k+1)}{\beta k}} = \frac{(\frac{\alpha}{\beta})^j C_N^j}{1 + \sum_{k=1}^N (\frac{\alpha}{\beta})^k C_N^k} = \frac{(\frac{\alpha}{\beta})^j C_N^j}{(1 + \frac{\alpha}{\beta})^N}.$$

(Here C_N^j is a notation for "N choose j".)