Probability III – 2008/09

Exercise Sheet 8

YOU ARE NOT SUPPOSED TO SUBMIT SOLUTIONS TO THIS CW. BUT YOU ARE STRONGLY ADVISED TO SOLVE THESE PROBLEMS AND TO COMPARE YOUR SOLUTIONS WITH THE MODEL SOLUTIONS.

Marks

[20] 1. A new product (a "Home Helicopter" to solve the commuting problem) is being introduced. The sales are expected to be determined by both media (newspaper and television) advertising and word-of-mouth advertising, wherein satisfied customers tell others about the product. Assume that media advertising creates new customers according to a Poisson process of rate $\alpha = 1$ customer per month. For the word-of-mouth advertising, assume that each purchaser of a Home Helicopter will generate sales to new customers at rate of $\theta = 2$ customers per month. Let $X(t)$ be the total number of Home Helicopters customers up to time t.

   (a) Model $X(t)$ as a pure birth process by specifying the birth parameters λ_k, for $k = 0, 1, \ldots$.

   (b) What is the probability that exactly two Home Helicopters are sold during the first two months.

[25] 2. Consider a population consisting of N individuals initially. There is no reproduction in this population. The individuals are subject to a mortality process. Their lifetimes T_k, $k = 1, 2, \ldots, N$, are mutually independent random variables distributed exponentially with mean θ. The time T until the population becomes extinct is a random variable.

   (a) Obviously, $T = \max(T_1, \ldots, T_N)$. Using this relation (or any other way of arguing) obtain the distribution function for T.

   (b) Let $X(t)$ be the number of individuals who are alive at time t. What are the parameters μ_k of this pure death process?

   (c) Show that

$$E(T) = \theta \left[\frac{1}{N} + \frac{1}{N-1} + \ldots + \frac{1}{1} \right]$$

   Hint. This example has been discussed in lectures. Note that

$$T = S_N + ... + S_1,$$

   where S_i is the time the process spends in state i (in the lecture we used the term the sojourn times). You are supposed to know the distribution of the random variable S_i.

[30] 3. Obtain the forward Kolmogorov differential equations for a birth and death process on a *finite* number $N+1$ of states. Use the Chapman-Kolmogorov relations

$$P_{ij}(t+s) = \sum_{k=0}^{N} P_{ik}(t)P_{kj}(s) = \sum_{k=0}^{N} P_{ik}(s)P_{kj}(t)$$

and the definition of the birth and death process. (Note that if $\{0, 1, ..., N\}$ are the states of the process, then $\mu_0 = 0$ and $\lambda_N = 0$.)

[25] 4. Consider a birth and death process on a *finite* state space $\{0, 1, ..., N\}$. Suppose that $\lambda_i > 0$ if $i \neq N$ and $\mu_i > 0$ if $i \neq 0$.

(a) Obtain the invariant (equilibrium) distribution for this process.

(b) Obtain the invariant distribution for the case when the death and birth parameters are $\lambda_n = \alpha(N-n)$, $\mu_n = \beta n$ (here $\alpha > 0,\ \beta > 0$).

Hint. This problem has been solved in the lecture in the case of an infinite state space. You are supposed to adjust the approach explained in the lecture to the case of a finite number of states.