Probability III – 2008/09

Solutions to Exercise Sheet 7

In lectures, we proved several statements about the Birth Process (BP). They can be briefly summarized as follows.

Theorem 1. *Suppose that $X(t)$ is a birth process with $X(0) = 0$. Set $p_n(t) \stackrel{\text{def}}{=} P\{X(t) = n|X(0) = 0\}$ Then the functions $p_n(t)$ satisfy the following equations:*

$$\begin{cases} p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t) & n \geq 0 \\ p_0(0) = 1 & \\ p_n(0) = 0 & \text{if } \; n > 0. \end{cases} \tag{1}$$

Theorem 2. *Equations (1) have a unique solution which can be found recursively using the following formulae:*

$$\begin{cases} p_0(t) = e^{-\lambda_0 t} \\ p_n(t) = \lambda_{n-1} e^{-\lambda_n t} \int_0^t e^{\lambda_n s} p_{n-1}(s) ds \quad n > 0 \end{cases} \tag{2}$$

Marks

[60] 1. Let $X(t)$ be a Birth Process with parameters $\lambda_0, \lambda_1, ..., \lambda_n, ...$ and suppose that $X(0) = 3$. Set $p_n(t) \stackrel{\text{def}}{=} P\{X(t) = n|X(0) = 3\}$

(a) Using the method explained in lectures derive equations for $p_3(t)$ and for $p_4(t)$.

Solution

Since $X(0) = 3$, state 3 plays in our case the same role as 0 in the case when $X(0) = 0$. The only difference is that now the parameters of the Birth Process are $\lambda_3, \lambda_4, ...$ (remark that $\lambda_0,\ \lambda_1,\ \lambda_2$ are unimportant because $X(t) \geq 3$). We can thus claim by analogy that

$$\begin{cases} p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t) & n \geq 3 \\ p_3(0) = 1 & \\ p_n(0) = 0 & \text{if } \; n > 3. \end{cases} \tag{3}$$

In particular, for $n = 3$ and taking into account that $p_2(t) \equiv 0$ we claim that

$$p_3'(t) = -\lambda_3 p_3(t), \quad \text{with} \quad p_3(0) = 1. \tag{4}$$

If $n = 4$ then the claim is that

$$p_4'(t) = -\lambda_4 p_4(t) + \lambda_3 p_3(t) \text{ with } p_4(0) = 0 \tag{5}$$

The proof is exactly the same as the one explained in lectures (with natural replacements of λ_0 by λ_3 etc. Nevertheless you are required to write it down. So here it is.

Proof of (4). For $h > 0$, the following equality follows from the monotonicity of the Birth Process (if $X(0) = 3$ and $X(t+h) = 3$, then also $X(t) = 3$)

$$\begin{aligned} P\{X(t+h) = 3|X(0) = 3\} &= P\{X(t) = 3, X(t+h) = 3|X(0) = 3\} \\ &= P\{X(t) = 3|X(0) = 3\}P\{X(t+h) = 3|X(t) = 3,\ X(0) = 3\}. \end{aligned} \tag{6}$$

By the Markov property

$$P\{X(t+h) = 3|X(t) = 3,\ X(0) = 3\} = P\{X(t+h) = 3|X(t) = 3\}.$$

Equation (6) can now be written as

$$p_3(t+h) = p_3(t)P\{X(t+h) = 3|X(t) = 3\} = p_3(t)(1 - \lambda_3 h + o(h)), \tag{7}$$

where the quality $P\{X(t+h) = 3|X(t) = 3\} = 1 - \lambda_3 h + o(h)$ is due to the definition of the Birth process. Equation (7) can be written as

$$p_3(t+h) = p_3(t) - \lambda_3 h p_3(t) + o(h) \quad \text{remember that } p_3(t)o(h) = o(h).$$

Hence

$$\frac{p_3(t+h) - p_3(t)}{h} = -\lambda_3 p_3(t) + \frac{o(h)}{h}.$$

Letting $h \to 0$ implies

$$p_3'(t) = -\lambda_3 p_3(t),$$

and this finishes the proof of (4). Finally, the initial condition $p_3(0) = 1$ follows from $p_3(0) = P\{X(0) = 3|X(0) = 3\} = 1$ which is clearly true.

Proof of (5). For $h > 0$, the following equality follows from the monotonicity property of the BP and the total probability formula:

$$\begin{aligned} &P\{X(t+h) = 4|X(0) = 3\} = \\ &P\{X(t) = 3|X(0) = 3\}P\{X(t+h) = 4|X(t) = 3,\ X(0) = 3\}+ \\ &P\{X(t) = 4|X(0) = 3\}P\{X(t+h) = 4|X(t) = 4,\ X(0) = 3\}. \end{aligned} \tag{8}$$

By the Markov property

$$\begin{aligned} P\{X(t+h) = 4|X(t) = 3,\ X(0) = 3\} &= P\{X(t+h) = 4|X(t) = 3\}, \\ P\{X(t+h) = 4|X(t) = 4,\ X(0) = 3\} &= P\{X(t+h) = 4|X(t) = 4\} \end{aligned}$$

Equation (8) can now be written as

$$p_4(t+h) = p_3(t)P\{X(t+h) = 4|X(t) = 3\} + p_4(t)P\{X(t+h) = 4|X(t) = 4\}. \tag{9}$$

Since by the definition of the Birth Process $P\{X(t+h)=4|X(t)=3\} = \lambda_3 h + o(h)$ and $P\{X(t+h)=4|X(t)=4\} = 1-\lambda_4 h + o(h)$ we obtain

$$p_4(t+h) = p_4(t)(1-\lambda_4 h + o(h)) + p_3(t)(\lambda_3 h + o(h)).$$

Hence

$$\frac{p_4(t+h)-p_4(t)}{h} = -\lambda_4 p_4(t) + \lambda_3 p_3(t) + \frac{o(h)}{h}.$$

Letting $h \to 0$ implies that

$$p_4'(t) = -\lambda_4 p_4(t) + \lambda_3 p_3(t),$$

and this finishes the proof of (5). The initial condition $p_4(0) = 0$ follows from $p_4(0) = P\{X(0)=4|X(0)=3\} = 0$.

(b) State the equation for $p_n(t)$.

Solution It is the same equation as before, namely

$$p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t). \tag{10}$$

The difference is that now $n \geq 4$ and that $p_3(t) = e^{-\lambda_3 t}$ (obviously, $p_i(t) \equiv 0,\ 0 \leq i \leq 2$).

(c) Derive formulae similar to (2) for $p_3(t)$ and for $p_n(t),\ n \geq 4$.

Solution For $n = 3$, it follows from (4) that $p_3(t) = ce^{-\lambda_3 t}$. Since $p_3(0) = 1$, we have $ce^{-\lambda_3 \times 0} = 1$. Hence $c = 1$ and $p_3(t) = e^{-\lambda_3 t}$.

If $n \geq 4$ then multiplying equation (10) by $e^{\lambda_n t}$ and writing it as

$$e^{\lambda_n t} p_n'(t) + e^{\lambda_n t} \lambda_n p_n(t) = \lambda_{n-1} e^{\lambda_n t} p_{n-1}(t), \quad n \geq 0$$

we obtain

$$\left(e^{\lambda_n t} p_n(t)\right)' = \lambda_{n-1} e^{\lambda_n t} p_{n-1}(t), \quad n \geq 3.$$

(make sure that you understand this calculation!). Hence

$$\int_0^t \left(e^{\lambda_n s} p_n(s)\right)' ds = \lambda_{n-1} \int_0^t e^{\lambda_n s} p_{n-1}(s) ds, \quad n \geq 3$$

and we obtain that

$$e^{\lambda_n t} p_n(t) - p_n(0) = \lambda_{n-1} \int_0^t e^{\lambda_n s} p_{n-1}(s) ds, \quad n \geq 3.$$

Thus

$$p_n(t) = e^{-\lambda_n t} p_n(0) + e^{-\lambda_n t} \lambda_{n-1} \int_0^t e^{\lambda_n s} p_{n-1}(s) ds, \quad n \geq 3.$$

If $n = 3$ then $p_3(0) = P\{X(0) = 3|X(0) = 3\} = 1$, $p_2(s) \equiv 0$ and the last formula implies that $p_3(t) = e^{-\lambda_3 t}$ (which means that we got one more derivation of the expression for $p_3(t)$). For $n > 3$, we have $p_n(0) = P\{X(0) = n|X(0) = 3\} = 0$ and

$$p_n(t) = e^{-\lambda_n t} \lambda_{n-1} \int_0^t e^{\lambda_n s} p_{n-1}(s) ds.$$

(d) Suppose that $\lambda_3 = 1$, $\lambda_4 = 1.5$. Find the expressions for $p_3(t)$ and for $p_4(t)$.

Solution We have just shown (this can also be inferred from (4)) that $p_3(t) = e^{-\lambda_3 t}$. Next, for $n = 4$ we have

$$p_4(t) = \lambda_3 e^{-\lambda_4 t} \int_0^t e^{\lambda_4 s} p_3(s) ds = \lambda_3 e^{-\lambda_4 t} \int_0^t e^{(\lambda_4 - \lambda_3)s} ds = \frac{\lambda_3}{\lambda_4 - \lambda_3}(e^{-\lambda_3 t} - e^{-\lambda_4 t}).$$

Hence the answer:

$$p_3(t) = e^{-t}, \quad p_4(t) = 2(e^{-t} - e^{-1.5t}).$$

(e) Find the probability density function of the time W_4 the process $X(t)$ remains in the state 3.

Solution The distribution function of W_4 is given by

$$F_{W_4}(y) = P\{W_4 \le y | X(0) = 3\} = 1 - P\{W_4 > y | X(0) = 3\}.$$

But $P\{W_4 > y | X(0) = 3\} = P\{X(y) = 3 | X(0) = 3\} = e^{-y}$ and hence $F_{W_4}(y) = 1 - e^{-y}$ and the p. d. f. is given by $f_{W_4}(y) = (1 - e^{-y})' = e^{-y}$ (note that $y \ge 0$ since $W_4 \ge 0$).

(f) Hence find the mean time $E(W_4 | X(0) = 3)$ the process $X(t)$ spends in the state 3.

Solution $E(W_4 | X(0) = 3) = \int_0^\infty y e^{-y} dy = 1.$

[40] 2. Consider the birth process (which we briefly discussed in lectures) with $\lambda_n = n\lambda$ ($\lambda > 0$).

(a) Prove that equations (1) imply that $P\{X(t) = 0 | X(0) = 0\} = 1$.

Proof. Note that $\lambda_0 = 0$. By the definition of $p_n(t)$ we have that $p_0(t) = P\{X(t) = 0 | X(0) = 0\} = e^{\lambda_0 t} = e^{0 \times t} = 1$.

(b) If, however, $X(0) = 1$ then

$$p_n(t) \stackrel{\text{def}}{=} P\{X(t) = n | X(0) = 1\} = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}. \qquad (11)$$

Check that this is true.

Hint: since you are given the explicit expressions for $p_n(t)$, it suffices to show that these functions satisfy equations (1) (and relevant initial conditions).

Solution Let us find the derivative of $p_n(t)$:

$$p_n'(t) = -\lambda e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} + \lambda(n-1)e^{-2\lambda t}(1 - e^{-\lambda t})^{n-2}.$$

Hence

$$p_n'(t) = -n\lambda e^{-\lambda t}(1-e^{-\lambda t})^{n-1} + (n-1)\lambda e^{-\lambda t}(1-e^{-\lambda t})^{n-1} + \lambda(n-1)e^{-2\lambda t}(1-e^{-\lambda t})^{n-2}.$$

In other words,

$$p_n'(t) = -\lambda_n p_n(t) + (n-1)\lambda e^{-\lambda t}(1 - e^{-\lambda t})^{n-2}(1 - e^{-\lambda t} + e^{-\lambda t})$$

and thus

$$p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t).$$

Since $p_n(t)$ defined by (11) also satisfies $p_1(0) = 1$ and $p_n(0) = 0$ for $n > 1$ we conclude that $p_n(t) = P\{X(t) = n|X(0) = 1\}$. (We use here the fact that these probabilities satisfy equations (1) with appropriate initial conditions and that there is only one sequence of of functions satisfying these equations.)

(c) The statement made in (b) means that the random variable $X(t)$ conditioned on $X(0) = 1$ has a geometric distribution. Hence, find $E(X(t)|X(0) = 1)$. Does $E(X(t)|X(0) = 1)$ grow exponentially fast as a function of t?

Solution

$$E(X(t)|X(0) = 1) = \sum_{n=1}^{\infty} n p_n(t) = \sum_{n=1}^{\infty} e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} = \frac{1}{e^{-\lambda t}} = e^{\lambda t}.$$

The answer is YES, this expectation grows exponentially fast.