# Probability III – 2008/09

## Solutions to Exercise Sheet 5

1. Let us agree that the state space of the chain is $S = \{1, 2, 3, 4, 5, 6\}$.

   **Transient and recurrent states.** State 6 is absorbing and therefore recurrent.

   States 3 and 5 form an equivalence class. This class is absorbing since the only allowed transition from state 3 is to state 5 and the only allowed transition from state 5 is to state 2. Once in one of these two states, the process will oscillate between them indefinitely (and periodically). Therefore states 2 and 4 are recurrent.

   Remember that, by definition, a state $i$ is transient if the probability for a MC starting from $i$ not to return to $i$ is positive. Note now that

   $$P\{X_n \text{ never returns to } 1|X_0 = 1\} \geq p_{16} = \frac{1}{3} > 0,$$

   $$P\{X_n \text{ never returns to } 2|X_0 = 2\} \geq p_{21}p_{16} = \frac{1}{6} > 0,$$

   $$P\{X_n \text{ never returns to } 4|X_0 = 4\} \geq p_{46} = \frac{1}{4} > 0$$

   and these inequalities imply that the states $1, 2, 4$ are *transient*. (In other words, we used the fact that the pathes $1 \to 6$, $2 \to 1 \to 6$, and $4 \to 6$ have positive probabilities which means that the MC starting from 1, 2 or 4 will be trapped in 6 before returning to 1, 2, 4 respectively.)

   **Irreducibility.** Remember that a MC is irreducible if all states of this chain intercommunicate with each other. Since state 6 communicates only with itself, the chain is *not irreducible*.

   (We also saw that each of the two states $\{3, 5\}$ is not intercommunicating with any of the states $1, 2, 4, 6$ which gives another proof of the fact that this MC is reducible.)

   **Equilibrium distributions.** A finite MC **always** has at least one equilibrium distribution (however, it may not have a limiting distribution!).

   As usual, the equilibrium distribution $\underline{w} = (w_1, w_2, w_3, w_4, w_5, w_6)$ can be found from the equations $\underline{w}\mathbf{P} = \underline{w}$ and $\sum_{j=1}^{6} w_j = 1$. By solving these equations (do it!) we obtain that $\underline{w} = (0, 0, a, 0, a, b)$, where $a$, $b$ are any real numbers such that $a \geq 0$, $b \geq 0$, $2a + b = 1$. The chain thus has infinitely many equilibrium distributions.

   E. g., the "simplest" one is given by $\underline{w} = (0, 0, 0, 0, 0, 1)$.

   **Exercises 1.** Prove, using the first step analysis, that the probability of absorbtion for a process starting from one of these states is 1.

   Hint. For instance, if we put

   $$a_i = P\{\text{the MC } X(t) \text{ would sometime reach 2, or 4, or 5 . } | \, X(0) = i\},$$

then it followis from the first step analysis that $a_0 = \frac{1}{3}a_0 + \frac{2}{3}$, and hence $a_0 = 1$. Similarly, $a_1 = \frac{1}{2}a_0 + \frac{1}{4}a_1 + \frac{1}{4}$ and $a_3 = \frac{1}{4}a_0 + \frac{1}{4}a_1 + \frac{1}{2}$. Hence $a_1 = a_3 = 1$.

**Exercise 2.** Let us change our matrix slightly:

$$\begin{Vmatrix} 1/3 & 0 & 1/3 & 0 & 0 & 1/3 \\ 1/2 & 1/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0.7 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 0.2 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

Which states are recurrent in this case? What is the difference between the behaviour of this MC and the one considered in the problem?

2. (i) The state 1 is recurrent because it is absorbing.

The states 2 and 3 are transient because

$$P\{\text{not to return to } 2|X_0 = 2\} \geq p_{21} = 0.3 > 0,$$

$$P\{\text{not to return to } 3|X_0 = 3\} \geq p_{31} = 0.4 > 0.$$

(ii) (a) We are going to use the following formulae which were derived in lectures:

$$f_i^{(n-1)} = p_{ii}^{(n)} - \sum_{k=1}^{n-1} f_i^{(k)} p_{ii}^{(n-k)}.$$

To do this , we have to calculate $p_{ii}^{(n)}$ for $i = 2$ and $n$ taking values 1, 2, 3. In other words, we have to find the relevant elements of $\mathbb{P}^2$ and $\mathbb{P}^3$. Here is the calculation:

$$\mathbb{P}^2 = \begin{Vmatrix} 1 & 0 & 0 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \end{Vmatrix} \begin{Vmatrix} 1 & 0 & 0 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 0 \\ 0.55 & 0.17 & 0.28 \\ * & * & * \end{Vmatrix}$$

and

$$\mathbb{P}^3 = \mathbb{P}^2\mathbb{P} = \begin{Vmatrix} 1 & 0 & 0 \\ 0.55 & 0.17 & 0.28 \\ * & * & * \end{Vmatrix} \begin{Vmatrix} 1 & 0 & 0 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 0 \\ * & 0.107 & * \\ * & * & * \end{Vmatrix}$$

(Remark that we do only the necessary calculations!). Hence

$$f_2^{(1)} = p_{22} = 0.3,$$
$$f_2^{(2)} = p_{22}^{(2)} - f_2^{(1)} p_{22}^{(1)} = 0.17 - 0.09 = 0.08,$$
$$f_2^{(3)} = p_{22}^{(3)} - f_2^{(1)} p_{22}^{(2)} - f_2^{(2)} p_{22}^{(1)} = 0.107 - 0.3 \times 0.17 - 0.08 \times 0.3 = 0.032.$$

(b) As the hint suggests, we consider the event

$$B = \{X_n \text{ will reach 2 before being absorbed by 1}\}$$

and note that, by the FSA, we have

$$\begin{aligned}\beta_2 =& \sum_{j=1}^{3} p_{2,j} P\{B|X_1 = j\} = p_{2,2} P\{B|X_1 = 2\} + p_{2,3} P\{B|X_1 = 3\} \\ =& p_{2,2}\beta_2 + p_{2,3} P\{B|X_0 = 3\}.\end{aligned}$$

Hence $\beta_2 = p_{2,2}\beta_2 + p_{2,3} P\{B|X_0 = 3\}$ and

$$\beta_2 = \frac{p_{2,3}}{1 - p_{2,2}} P\{B|X_0 = 3\}$$

Next, set $u_i = P\{B|X_0 = i\}$. Then by the FSA

$$\begin{aligned}u_1 =& 0 \\ u_2 =& 1 \\ u_3 =& \sum_{j=1}^{3} p_{3,j} u_j = 0.2 + 0.4 u_3\end{aligned}$$

and hence $u_3 = \frac{1}{3}$ and $\beta_2 = \frac{p_{2,3}}{1-p_{2,2}} \times \frac{1}{3} = \frac{4}{21}$.

(iii) As we know, $\beta_2 = \frac{U}{1+U}$ and hence $U = \frac{\beta_2}{1-\beta_2} = \frac{4}{17}$.

(iv) We proved in lectures (see also notes) that $E(M) = \frac{\beta_2}{1-\beta_2} = \frac{4}{17}$.

3. It is evident from the transition matrix that $0 \leftrightarrow 2$, $4 \leftrightarrow 5$, and $1 \leftrightarrow 3$. Since $p_{00} + p_{04} = 1$ and $p_{01} + p_{44} = 1$, it is impossible to reach from any of the states $\{0, 2\}$ any of the other states. Similarly, it is impossible to reach from any of the states $\{4, 5\}$ any of the other states.

   Thus, the state space is partitioned into three equivalence classes: $\{0, 2\}$, $\{4, 5\}$, and $\{1, 3\}$.

   Note that $\{0, 2\}$, $\{1, 3\}$ are two classes of recurrent states whereas $\{4, 5\}$ is a transient class. Why?