Probability III – 2008/09

Exercise Sheet 5

Write your name and student number at the top of your assignment before handing it in. Staple all pages together. Return the assignment by 16:00 on Thursday, 5 March

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[25] 1. Which states are transient and which are recurrent in the Markov chain whose transition probability matrix is

$$\left\| \begin{array}{cccccc} 1/3 & 0 & 1/3 & 0 & 0 & 1/3 \\ 1/2 & 1/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\| ?$$

Explain why. Is this chain irreducible? Does it have an equilibrium distribution?

[50] 2. A Markov chain has state space $S = \{1, 2, 3\}$ and transition probability matrix

$$\mathbb{P} = \left\| \begin{array}{ccc} 1 & 0 & 0 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \end{array} \right\|$$

(i) Explain briefly which of the states are recurrent and which are transient.

(ii) Calculate the following probabilities:

(a) $f_2^{(n)} = P\{X_n = 2,\ X_k \neq 2 \text{ for } k = 1, .., n-1 | X_0 = 2\}$ for $n = 1, 2, 3$.

(b)

$$\beta_2 = P\{X_n \text{ will return to } 2 | X_0 = 2\}$$

Hint. Consider the event $B = \{X_n \text{ will reach } 2 \text{ before being absorbed by } 1\}$. Note that, by the FSA, we have

$$\begin{aligned} \beta_2 =& \sum_{j=1}^{3} p_{2,j} P\{B|X_1 = j\} = p_{2,2} P\{B|X_1 = 2\} + p_{2,3} P\{B|X_1 = 3\} \\ =& p_{2,2}\beta_2 + p_{2,3} P\{B|X_0 = 3\}. \end{aligned}$$

You will have to use FSA in order to find $P\{B|X_0 = 3\}$.

(iii) Use the relation between β_2 and $U = \sum_{n=1}^{\infty} p_{22}^{(n)}$ derived in lectures and find U.

(iv) Use β_2 and find $E(M)$ where M is the number of returns to 2.

[25] 3. A Markov chain on states 0,1,2,3,4, and 5 has the following transition probability matrix:

$$\left\| \begin{array}{cccccc} 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 4/5 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{array} \right\| .$$

Find all communicating (equivalence) classes.