Probability III – 2008/09

Solutions to Exercise Sheet 4

1. Write $\mathbb{P}$ as

$$\mathbb{P} = \left\| \begin{array}{ccccc} 0 & + & + & 0 & 0 \\ + & 0 & + & 0 & 0 \\ + & + & 0 & + & 0 \\ 0 & 0 & + & 0 & + \\ + & 0 & 0 & + & 0 \end{array} \right\|$$

Then

$$\mathbb{P}^2 = \left\| \begin{array}{ccccc} + & + & + & + & 0 \\ + & + & + & + & 0 \\ + & + & + & 0 & + \\ + & + & 0 & + & 0 \\ 0 & + & + & 0 & + \end{array} \right\|$$

$$\mathbb{P}^4 = \left\| \begin{array}{ccccc} + & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & 0 \\ + & + & + & + & + \end{array} \right\|$$

and

$$\mathbb{P}^5 = \mathbb{P}\mathbb{P}^4 = \left\| \begin{array}{ccccc} + & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & + \end{array} \right\|$$

Therefore $\mathbb{P}$ is regular.

The π_i, $i = 0, 1, 2, 3, 4$, satisfy the equations

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

$$\begin{aligned} \pi_0 &= \pi_1/2 + \pi_2/3 + \pi_4/2 \\ \pi_1 &= \pi_0/2 + \pi_2/3 \\ \pi_2 &= \pi_0/2 + \pi_1/2 + \pi_3/2 \\ \pi_3 &= \pi_2/3 + \pi_4/2 \\ \pi_4 &= \pi_3/2 \end{aligned}$$

We can solve, for example, the equations

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

$$\begin{aligned}
\pi_1 &= \pi_0/2 + \pi_2/3 \\
\pi_2 &= \pi_0/2 + \pi_1/2 + \pi_3/2 \\
\pi_3 &= \pi_2/3 + \pi_4/2 \\
\pi_4 &= \pi_3/2
\end{aligned}$$

We will solve for π_0, π_1, π_2 and π_4 in terms of π_3; the rest is easy.

We already have $\pi_4 = \pi_3/2$. The fourth equation gives $\pi_2/3 = \pi_3 - \pi_4/2 = \pi_3 - \pi_3/4$ or $\pi_2 = 9\pi_3/4$. Subtracting the third equation from the second and solving for π_1 gives $\pi_1 = 8\pi_2/9 - \pi_3/3 = 5\pi_3/3$. Solving for π_0 in the first equation gives $\pi_0 = 2\pi_1 - 2\pi_2/3 = 11\pi_3/6$.

Substituting these expressions into the first equation gives

$$\pi_3 \left(\frac{11}{6} + \frac{5}{3} + \frac{9}{4} + 1 + \frac{1}{2} \right) = 1$$

so $\pi_3 = \frac{12}{87} = \frac{4}{29}$. The expressions give $\pi_0 = \frac{22}{87}, \pi_1 = \frac{20}{87}, \pi_2 = \frac{9}{29}, \pi_4 = \frac{2}{29}$.

2. By solving the equations

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\begin{aligned}
\pi_0 &= 0.3\pi_0 + 0.2\pi_1 + 0.4\pi_2 \\
\pi_1 &= 0.5\pi_0 + 0.7\pi_1 + 0.2\pi_2
\end{aligned}$$

one finds that $\pi_0 = \frac{16}{61}, \pi_1 = \frac{34}{61}, \pi_2 = \frac{11}{61}$.

Now,

$$\begin{aligned}
\lim_{n\to\infty} P(X_{n-1} = 1 | X_n = 2) &= \lim_{n\to\infty} \frac{P(X_{n-1} = 1, X_n = 2)}{P(X_n = 2)} \\
&= \lim_{n\to\infty} \frac{P(X_{n-1} = 1) P(X_n = 2 | X_{n-1} = 1)}{P(X_n = 1)} \\
&= \lim_{n\to\infty} \frac{P(X_{n-1} = 1) p_{1,2}}{P(X_n = 2)} \\
&= p_{1,2} \lim_{n\to\infty} \frac{P(X_{n-1} = 1)}{P(X_n = 2)} \\
&= \frac{p_{1,2}\pi_1}{\pi_2} \\
&= \frac{1}{10} \times \frac{34}{61} \times \frac{61}{11} = \frac{17}{55}
\end{aligned}$$

3. a) The equations are

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

$$\begin{aligned}
\pi_0 &= 0.2\pi_0 + 0.5\pi_1 + 0.2\pi_2 + 0.1\pi_3 \\
\pi_1 &= 0.2\pi_0 + 0.2\pi_1 + 0.3\pi_2 + 0.2\pi_3 \\
\pi_2 &= 0.4\pi_0 + 0.2\pi_1 + 0.4\pi_2 + 0.4\pi_3
\end{aligned}$$

Solving these equations gives $\pi_0 = \frac{13}{51}, \pi_1 = \frac{12}{51}, \pi_2 = \frac{18}{51}, \pi_3 = \frac{8}{51}$.

b) In the long run the probability that the process is out-of-control is $\pi_2 + \pi_3 = \frac{26}{51}$.

4. Let X_n be the number of balls in urn A. It is clear that the distribution of X_{n+1} depends only on the value of X_n (and does not depend on the values of $X_k,\ k \leq n-1$. Hence X_n is a MC. Since each urn can be chosen with probability 0.5, the transition probabilities are given by

$$p_{i,i+1} = p_{i,i-1} = \frac{1}{2} \text{ if } i = 1, 2, 3, 4; \quad p_{0,1} = p_{0,0} = p_{5,4} = p_{5,5} = \frac{1}{2}$$

or, equivalently, by

$$\mathbb{P} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

It is easy to check that the equilibrium distribution is given by $\underline{w} = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. Hence, for large values of time n, the fraction of time the MC spends in the state 0 (urn A is empty) is $\frac{n_0}{n} \simeq \frac{1}{6}$.