Probability III – 2008/09

Solutions to Exercise Sheet 3

1. This solution is very much in the spirit of the first solution to a similar problem discussed in the lectures and in the notes (Extended example). As there, we write 1 for H and 0 for T. In these notations the repeated flipping of the coin leads to the appearance of the sequence of patterns (ξ_0, ξ_1, ξ_2), (ξ_1, ξ_2, ξ_3),... which form a MC. We order all possible patterns as follows

$1 \leftrightarrow (101)$, $2 \leftrightarrow (100)$, $3 \leftrightarrow (001)$, $4 \leftrightarrow (000)$, $5 \leftrightarrow (011)$, $6 \leftrightarrow (010)$, $7 \leftrightarrow (111)$, $8 \leftrightarrow (110)$.

after which we can say that our MC X_n takes values in the state space $S = \{1, 2, ..., 8\}$. Since in our problem states 2 an 7 are absorbing, the transition matrix is given by

$$\mathbb{P} = \left\| \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right\|$$

and the initial distribution is $p_i \stackrel{\text{def}}{=} P\{X_0 = i\} = \frac{1}{8}$ (all patterns are equally likely to appear after the first three flips).
We now define a function f on S as follows:

$$f(i) = \begin{cases} -1 & \text{if } i = 1 \\ 1 & \text{if } i = 4 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$F = f(X_0) + f(X_1) + ... f(X_{T-1}) + f(X_T)$$

is the reward given to the player. Indeed, it is obvious that

$$F = (\text{number of appearances of TTT}) - (\text{number of appearances of HTT}).$$

Hence all we need to do is to find $E(F) = \sum_{i=1}^{m} p_i w_i$ where $w_i = E(F \,|\, X_0 = i)$. According to the general Theorem 3.1 from the notes concerning FSA, we have

$$\begin{cases} w_i = f(i) & \text{if } i \text{ is absorbing} \\ w_i = f(i) + \sum_{j=1}^{m} p_{ij} w_j & \text{if } i \text{ is not absorbing} \end{cases} \tag{1}$$

In our concrete setting equations (2) imply $w_2 = w_7 = 0$ (for absorbing states!) and

$$
\begin{aligned}
w_1 =& -1 + \frac{1}{2}(w_5 + w_6)\\
w_3 =& \frac{1}{2}(w_5 + w_6)\\
w_4 =& 1 + \frac{1}{2}(w_3 + w_4)\\
w_5 =& \frac{1}{2} w_8\\
w_6 =& \frac{1}{2} w_1\\
w_8 =& \frac{1}{2} w_1
\end{aligned}
\tag{2}
$$

Solving these equations gives $w_1 = -\frac{8}{5}$, $w_3 = -\frac{3}{5}$, $w_4 = \frac{7}{5}$, $w_5 = -\frac{2}{5}$, $w_6 = -\frac{2}{5}$, $w_8 = -\frac{2}{5}$. Finally the player's averaged gain is $E(F) = \frac{1}{8}\sum_{i=1}^{8} w_i = -\frac{7}{20}$. Negative gain means that on average the player looses money and hence should not take part in this game.

2. As usual, we describe the process as a sequence of independent identically distributed random variables $\xi_0, \xi_1, ..., \xi_n, ...$ with $P\{\xi_n = i\} = \frac{1}{6}$, where i takes integer values, $1 \le i \le 6$. The start of the game is denoted by 0 and the game stops when one of the patterns 112, 121, 211 appears for the first time. Set

$$S = \{0, 1, 2, 11, 12, 21, 112, 121, 211\}$$

The MC X_n can now be defined as follows. It starts with $X_0 = 0$. Next, $X_1 = \xi_0$ if $\xi_0 = 1$ or $\xi_0 = 2$ and $X_1 = 0$ if $3 \le \xi_0 \le 6$.

Let us explain the general rule for defining X_{n+1} when X_n and ξ_n are given. Rather than doing this formally, let us consider two examples.

(a) If $X_n = 1$ then $X_{n+1} = \begin{cases} 0 & \text{if } 3 \le \xi_n \le 6, \\ 11 & \text{if } \xi_n = 1, \\ 12 & \text{if } \xi_n = 2, \end{cases}$

(b) If $X_n = 21$ then $X_{n+1} = \begin{cases} 0 & \text{if } 3 \le \xi_n \le 6, \\ 211 & \text{if } \xi_n = 1, \\ 12 & \text{if } \xi_n = 2, \end{cases}$

All other transitions are defined similarly. One has to remember that it is necessary to define the transition from X_n only when X_n is not an absorbing state. Note that $P\{X_{n+1} = 0 | X_n = \text{non-absorbing state}\} = \frac{2}{3}$ because this event takes place if and only if $3 \le \xi_n \le 6$ (in other words, this probability does not depend on the actual value of a non-absorbing X_n). All other transition probabilities are either 0 or $\frac{1}{6}$.

Now, if the states are ordered as they are listed in S, then it is easy to see that the transition matrix of this MC is

$$\mathbb{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \\ \frac{2}{3} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3. This problem is similar to problem 3 from CW2. However, this time

$$S = \{0, 5, 6, (56), (65), (66)\}$$

and we have 3 absorbing states: (56), (65), (66). In order to be ale to write p_{ij}, $0 \le i, j \le 5$ rather than for instance $p_{5,(56)}$ we shall list them as follows: $S = (0, 1, 2, 3, 4, 5)$ with the one to one correspondence given by $0 \leftrightarrow 0$, $5 \leftrightarrow 1$, $6 \leftrightarrow 2$, $(56) \leftrightarrow 3$, $(65) \leftrightarrow 4$, $(66) \leftrightarrow 5$. The non-zero transition probabilities are given by

$$p_{00} = p_{10} = p_{20} = \frac{2}{3}, \;\; p_{01} = p_{02} = p_{11} = p_{13} = p_{24} = p_{25} = \frac{1}{6}, \;\; p_{33} = p_{44} = p_{55} = 1$$

(explain these numbers – this is easy!). Thus

$$\mathbb{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let X_n be a MC with the above transition matrix. The questions we are interested in can now be stated as follows:

(a) Obviously, the number of times the die is rolled is the same as the time of absorbtion T and our task is: find $E(T|X_0 = 0)$.

(b) The probability that the game will end when the sum of two last results is 12 is the same as $P\{X_T = 5|X_0 = 0\}$. Find this probability.

(c) Finally the question "Which of the two numbers is observed more often on average in this game, 5 or 6?" is equivalent to asking the following. Let a and h be the number of visits to 5 and 6 respectively. Which of the two averages is larger, $E(a|X_0 = 0)$ or $E(h|X_0 = 0)$?

We first answer (a). Set $v_i = E(T|X_0 = i)$. As has been explained above (also, see equations (6) from the notes on FSA), the v_i satisfy the equations $v_3 = v_4 = v_5 = 0$

$$v_0 = 1 + \frac{2}{3}v_0 + \frac{1}{6}v_1 + \frac{1}{6}v_2$$
$$v_1 = 1 + \frac{2}{3}v_0 + \frac{1}{6}v_1 + \frac{1}{6}v_3$$
$$v_2 = 1 + \frac{2}{3}v_0 + \frac{1}{6}v_4 + \frac{1}{6}v_5$$

We now easily find that $v_0 = \frac{123}{8} = 15.375$ (which is the answer), $v_1 = \frac{27}{2} = 13.5$, and $v_2 = \frac{45}{4} = 11.25$.
Let us now consider (b). Set $u_i = P\{X_T = 5|X_0 = i\}$. Then by the FSA we have $u_3 = u_4 = 0, u_5 = 1$ and

$$u_0 = \frac{2}{3}u_0 + \frac{1}{6}u_1 + \frac{1}{6}u_2$$
$$u_1 = \frac{2}{3}u_0 + \frac{1}{6}u_1 + \frac{1}{6}u_3$$
$$u_2 = \frac{2}{3}u_0 + \frac{1}{6}u_4 + \frac{1}{6}u_5$$

Solving now the equations gives the answer $u_0 = \frac{5}{16}$ (and $u_1 = \frac{1}{4}$, $u_2 = \frac{3}{8}$).

Finally, let us turn to (c). According to example 2 and equations (7) from FSA notes, we have to do the following.
Set $f(1) = 1$ and $f(i) = 0$ for $i \neq 1$. Remember that then $a = \sum_{j=0}^{T} f(X_j)$ and the $b_i \stackrel{\text{def}}{=} E(a|X_0 = i)$ satisfy the following equations: $b_3 = b_4 = b_5 = 0$ and

$$b_0 = \frac{2}{3}b_0 + \frac{1}{6}b_1 + \frac{1}{6}b_2$$
$$b_1 = 1 + \frac{2}{3}b_0 + \frac{1}{6}b_1 + \frac{1}{6}b_3$$
$$b_2 = \frac{2}{3}b_0 + \frac{1}{6}b_4 + \frac{1}{6}b_5$$

Solving these equations we obtain: $b_0 = \frac{9}{4}$ which is the answer ($b_1 = 3$, $b_2 = \frac{3}{2}$).
Let us now find $E(h|X_0 = 0)$. This time set $f(2) = 1$ and $f(i) = 0$ for $i \neq 2$. Then $h =$

$\sum_{j=0}^{T} f(X_j)$ and the $h_i \stackrel{\text{def}}{=} E(h|X_0 = i)$ satisfy the following equations: $h_3 = h_4 = h_5 = 0$ and

$$h_0 = \frac{2}{3}h_0 + \frac{1}{6}h_1 + \frac{1}{6}h_2$$
$$h_1 = \frac{2}{3}h_0 + \frac{1}{6}h_1 + \frac{1}{6}h_3$$
$$h_2 = 1 + \frac{2}{3}h_0 + \frac{1}{6}h_4 + \frac{1}{6}h_5$$

Solving these equations we obtain: $h_0 = \frac{15}{8}$ (and $h_1 = \frac{3}{2}$, $h_2 = \frac{9}{4}$).
Hence the average number of observed 5's is larger than the average number of observed 6's.

Remark. Note that in the solution given above a is the number of 5's observed strictly before the end of the game. In other words, if the game ends with $X_T = (65)$ then the last 5 is not counted. The same apples to h: if $X_T = (56)$ or $X_T = (66)$ then the last 6 is not counted. How should one amend the solution presented above in order to include the last appearing digit into the count?

Please let me know if you have any comments or corrections