Probability III – 2008/09

Solutions to Exercise Sheet 2

1. a) Take $S = \{0,1,2,3,4,5\}$ as the state space of the random process X_t with $X_t = i$ indicating that the discount is $10\%i$ at the end of the year t. We set $X_0 = 0$ as there is no discount initially. The X_t forms a MC since we can find the relevant transition probabilities. Namely, if $X_t = i$ then either $X_{t+1} = 0$ with probability p (the claim was made during the year $t+1$) or $X_{t+1} = \min(i+1,5)$ with probability $1-p$ (no claims during the year $t+1$).
b) We already know that $p_{i0} = p$ for any $i \in S$, $p_{55} = 1-p$, and $p_{i,i+1} = 1-p$ if $0 \le i \le 4$.
Hence

$$\mathbb{P} = \begin{pmatrix} p & 1-p & 0 & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 & 0 \\ p & 0 & 0 & 1-p & 0 & 0 \\ p & 0 & 0 & 0 & 1-p & 0 \\ p & 0 & 0 & 0 & 0 & 1-p \\ p & 0 & 0 & 0 & 0 & 1-p \end{pmatrix}$$

c) The probability that $X_4 = 2$ given that $X_0 = 0$ can be found by considering all paths of length 4 from state 0 to state 2. We obtain

$$P\{X_4 = 2 \,|\, X_0 = 0\} = p_{00}p_{00}p_{01}p_{12} + p_{01}p_{10}p_{01}p_{12} = p^2(1-p)^2 + (1-p)p(1-p)^2 = p(1-p)^2.$$

d) The answer is $P\{X_{16} = 0\} = p$. Indeed, we have

$$P\{X_{16}=0\} = \sum_{i=0}^{5} P\{X_{15}=i\}P\{X_{16}=0 \,|\, X_{15}=i\} = \sum_{i=0}^{5} P\{X_{15}=i\}p_{i0} = p\sum_{i=0}^{5} P\{X_{15}=i\} = p$$

since $p_{i0} = p$ for all i and $\sum_{i=0}^{5} P\{X_{15} = i\} = 1$.

2. This question is best tackled using first step analysis.
a) Let v_i be the expected time to absorbtion starting from state i. Writing T for the absorbtion time, we set

$$v_i = E(T|X_0 = i).$$

Using first step analysis (see lecture notes for more details) we obtain the equations

$$v_0 = \sum_{k=0}^{2} p_{0k}E(T|X_0 = 0,\ X_1 = k) = \sum_{k=0}^{2} p_{0k}(1 + E(T|X_0 = k)) = \sum_{k=0}^{2} p_{0k}(1 + v_k)$$

We use here that $E(T|X_0 = 0,\ X_1 = k) = 1 + E(T|X_1 = k)$ and $E(T|X_1 = k) = E(T|X_0 = k) = v_k$.
Similarly

$$v_1 = \sum_{k=0}^{2} p_{1k}E(T|X_0 = 0,\ X_1 = k) = \sum_{k=0}^{2} p_{1k}(1 + v_k)$$

Taking into account that $E(T|X_0 = 2) = 0$ and using the values p_{ij} from the transition matrix we obtain

$$v_0 = 1 + 0.2v_1$$
$$v_1 = 1 + 0.25v_0 + 0.4v_1.$$

These equations are easily solved to give $v_0 = \frac{16}{11}$ (and $v_1 = \frac{25}{11}$).
ii) Let w_i be the expected number of visits to state 1 before absorbtion starting from state i. We write T for the absorbtion time. We now use the general method described in lectures with $f(1) = 1$ and $f(0) = f(2) = 0$. Then the number of visits to 1 before absorbtion is given by

$$F = f(X_0) + f(X_1) + ... + f(X_T)$$

since the j-th term in this sum is 1 if $X_j = 1$ and is 0 otherwise. Then $w_i = E(F|X_0 = i)$ and equations for w_i are as follows:

$$w_0 = f(0) + p_{00}w_0 + p_{01}w_1 + p_{02}w_2$$
$$w_1 = f(1) + p_{10}w_0 + p_{11}w_1 + p_{12}w_2$$

Taking into account that $w_2 = 0$ (by the definition of this quantity), we obtain

$$w_0 = 0.2w_1$$
$$w_1 = 1 + 0.25w_0 + 0.4w_1$$

These equations are easily solved to give $w_0 = \frac{4}{11}$ (and $w_1 = \frac{20}{11}$).
iii) Let g_i be your expected gain up to absorbtion starting from state i. Using the same method but this time with $f(0) = -3$, $f(1) = 5$, $f(2) = 0$ we obtain

$$g_0 = -3 + 0.2g_1$$
$$g_1 = 5 + 0.25g_0 + 0.4g_1.$$

Solving these equation we obtain $g_0 = -\frac{16}{11}$ (which in fact is a loss whereas $g_1 = \frac{85}{11}$ could be a gain had our process started at 1).

3. Our state space $S = \{0, 5, 6, (56), (65)\}$. In order to be ale to write p_{ij}, $0 \le i, j \le 4$ rather than for instance $p_{5,(56)}$ we shall first of all change the notations for our states. Namely we set $S = (0, 1, 2, 3, 4, 5)$ where the one to one correspondence is as follows: $0 \leftrightarrow 0$, $5 \leftrightarrow 1$, $6 \leftrightarrow 2$, $(56) \leftrightarrow 3$, $(65) \leftrightarrow 4$. The non-zero transition probabilities are then given by

$$p_{00} = \frac{2}{3},\ p_{01} = p_{02} = \frac{1}{6},\ p_{10} = p_{20} = \frac{2}{3},\ p_{11} = p_{22} = \frac{1}{6},\ p_{13} = p_{24} = \frac{1}{6},\ p_{33} = p_{44} = 1$$

(explain these numbers – this is easy!). Thus

$$\mathbb{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{2}{3} & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ \frac{2}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let X_n be a MC with the above transition matrix. The questions we are interested in can now be stated as follows:
a) What is the expected time of absorbtion of our chain if $X_0 = 0$?
b) What is the probability that the chain starting from 0 will be absorbed by state 3 (rather than 4).
We first answer a). Set $v_i = E(T|X_0 = i)$ and $f(0) = f(1) = f(2) = 1$ and $f(3) = f(4) = 0$. We then have

$$T = f(X_0) + f(X_1) + ... + f(X_T)$$

and the v_i satisfy the equations

$$v_0 = 1 + \frac{2}{3}v_0 + \frac{1}{6}v_1 + \frac{1}{6}v_2$$
$$v_1 = 1 + \frac{2}{3}v_0 + \frac{1}{6}v_1 + \frac{1}{6}v_3$$
$$v_2 = 1 + \frac{2}{3}v_0 + \frac{1}{6}v_2 + \frac{1}{6}v_4$$

Since we know that $v_3 = v_4 = 0$ we easily find that $v_0 = 21$ (and $v_1 = v_2 = 18$).
Let us now consider b). Set $u_i = P\{X_T = 3|X_0 = i\}$ and note that if we choose this time f so that $f(0) = f(1) = f(2) = f(4) = 0$ and $f(3) = 1$, then

$$E(f(X_0) + f(X_1) + ... + f(X_T)|X(0) = i) = E(f(X_T)|X(0) = i) = P\{X_T = 3|X_0 = i\} = u_i$$

(we repeat, for this particular case, the argument which has been discussed in lectures and can be looked up in the notes concerning the FSA). We thus have

$$u_0 = \frac{2}{3}u_0 + \frac{1}{6}u_1 + \frac{1}{6}u_2$$
$$u_1 = \frac{2}{3}u_0 + \frac{1}{6}u_1 + \frac{1}{6}u_3$$
$$u_2 = \frac{2}{3}u_0 + \frac{1}{6}u_2 + \frac{1}{6}u_4$$

This time $u_3 = 1$, $u_4 = 0$. Solving now the equations gives $u_0 = \frac{1}{2}$, $u_1 = \frac{3}{5}$, $u_2 = \frac{2}{5}$.

Please let me know if you have any comments or corrections