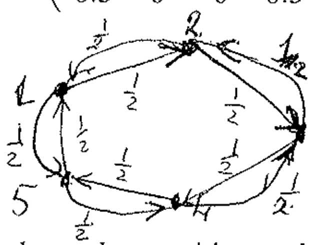
Probability III – 2008/09

Solutions to Exercise Sheet 1

1. This is a very easy exercise since all the p_{ij} are given. Thus

$$\mathbb{P} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \end{pmatrix}$$

The transition graph is



2. We start with X_n. Let's work out the transition probabilities for all pairs of states. It is clear that

$$\mathbb{P}(X_{n+1} = j | X_n = i) = 0 \qquad \text{if } i > j.$$

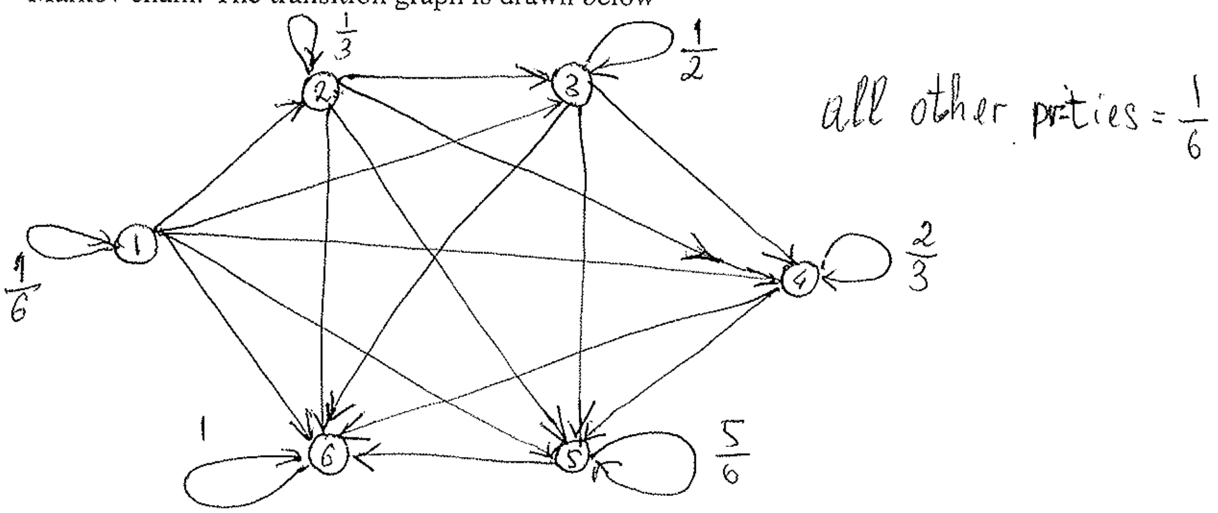
If $X_n = i$ then $X_{n+1} = i$ if and only if throw number $(n+1)$ is no more than i. Thus,

$$\mathbb{P}(X_{n+1} = i | X_n = i) = \frac{i}{6}.$$

If $i < j$, and $X_n = i$ then $X_{n+1} = j$ if and only if throw number $(n+1)$ is a j. Thus,

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \frac{1}{6} \qquad \text{if } i < j.$$

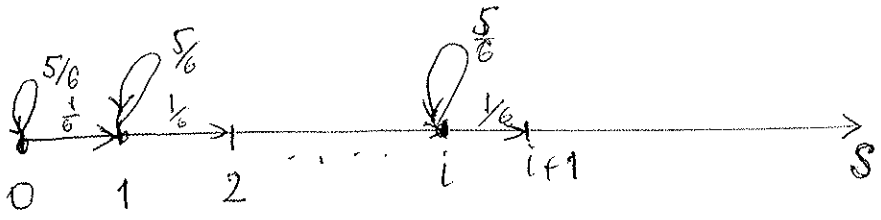
Since these transition probabilities do not depend on $X_{n-1}, X_{n-2}, \ldots$ we deduce that X_n is a Markov chain. The transition graph is drawn below



Next, we consider S_n. If throw number $(n+1)$ is not a 6 then he number of 6s seen on the first $(n+1)$ rolls is the number of sixes seen in the first n rolls. If throw number $(n+1)$ is a 6 then it increases by 1. Thus

$$\mathbb{P}(S_{n+1} = j | S_n = i) = \begin{cases} \frac{5}{6} & \text{if } i = j \\ \frac{1}{6} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Again, these transition probabilities do not depend on $S_{n-1}, S_{n-2}, \ldots$. We deduce that S_n is a Markov chain. The transition graph is drawn below



Finally, T_n is not a Markov chain. To show this we must show that the Markov condition is violated. Suppose that $T_1 = 1, T_2 = 1$; in this case the first throw must have been a 6 and the second throw must have been something else. Thus T_3 is at most 1. It follows that

$$\mathbb{P}(T_3 = 2 | T_2 = 1, T_1 = 1) = 0.$$

Suppose that $T_1 = 0, T_2 = 1$; in this case the first throw must have been something other than a 6 and the second throw must have been a 6. Thus, if the third throw is a 6 then $T_3 = 2$. It follows that,

$$\mathbb{P}(T_3 = 2 | T_2 = 1, T_1 = 0) = \frac{1}{6}.$$

We deduce that $\mathbb{P}(T_3 = 2 | T_2 = 1)$ does depend on T_1 and hence the Markov property does not hold.

3. Note that the matrix $\mathbb{P}$ has the following property:

$$\sum_{i=1}^{3} p_{ij} = 1 \text{ for all } j,\ 1 \leq j \leq 3.$$

Hence

$$P\{Y_1 = j\} = \sum_{i=1}^{3} P\{Y_0 = i\} P\{Y_1 = j \,|\, Y_0 = i\} = \sum_{i=1}^{n} p_i p_{ij} = \frac{1}{3} \sum_{i=1}^{n} p_{ij} = \frac{1}{3}$$

To finish the proof, we use induction. If we already know that $P\{Y_n = j\} = \frac{1}{3}$ then

$$P\{Y_{n+1} = j\} = \sum_{i=1}^{3} P\{Y_n = i\} P\{Y_{n+1} = j \,|\, Y_n = i\} = \sum_{i=1}^{3} \frac{1}{3} p_{ij} = \frac{1}{3}. \quad \square$$

Generalization. Replacing 3 by any m in the above formulae would not change anything in the proof. The fact that we have actually proved is that if $P\{Y_0 = j\} = \frac{1}{m}$ and

$$\sum_{i=1}^{m} p_{ij} = 1 \text{ for all } j,\ 1 \leq j \leq m.$$

then $P\{Y_n = j\} = \frac{1}{m}$ for all $n \geq 1$ and all j, $1 \leq j \leq m$.

4. Observe that rules of the game are such that the number of green balls is always equal to the number of red balls. If $X_n = i$ then either $X_{n+1} = i$ or $X_{n+1} = i-1$. We now have for $i = 0,1,2,3,4$, that

$$P\{X_{n+1} = i-1 \,|X_n = i\} = P\{\text{ choose one of } i \text{ red and one of } i \text{ green balls }\} = \frac{i^2}{\binom{8}{2}} = \frac{i^2}{28}$$

and hence

$$P\{X_{n+1} = i \,|X_n = i\} = 1 - \frac{i^2}{28}.$$

The transition matrix $(P_{ij})_{0\leq i,j\leq 4}$ now looks as follows

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{28} & \frac{27}{28} & 0 & 0 & 0 \\ 0 & \frac{1}{7} & \frac{6}{7} & 0 & 0 \\ 0 & 0 & \frac{9}{28} & \frac{19}{28} & 0 \\ 0 & 0 & 0 & \frac{4}{7} & \frac{3}{7} \end{pmatrix}$$

5. If $X_n = i$, $1 \leq i \leq 4$, then either $X_{n+1} = i$ or $X_{n+1} = i+1$. It is easy to find $P\{X_{n+1} = i+1 \,|X_n = i\}$. Let A_i be the event that one of the interacting people is diseased and the other one is not. Then $P(A_i) = \frac{i(4-i)}{6}$ and

$$P\{X_{n+1} = i+1 \,|X_n = i\} = P(A_i)\,0.2 = 0.2\,\frac{i(4-i)}{6}.$$

Hence $P\{X_{n+1} = i \,|X_n = i\} = 1 - 0.2\,\frac{i(4-i)}{6}$ and we have

$$\mathbb{P} = \begin{pmatrix} 0.9 & 0.1 & 0 & 0 \\ 0 & 1-0.2\frac{1}{3} & 0.2\frac{1}{3} & 0 \\ 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Please let me know if you have any comments or corrections