Probability III – 2008/09

Exercise Sheet 1

Write your name and student number at the top of your assignment before handing it in. Staple all pages together. Return the assignment to me by 17:00 on Monday, 19 January

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1. Let X_n be a Markov chain with a state space $S = (1,2,3,4,5)$ and transition probabilities [10]

$$p_{i,i\pm1} = 0.5 \text{ if } 2 \leq i \leq 4 \text{ and } p_{1,2} = p_{1,5} = p_{5,4} = p_{5,1} = 0.5.$$

Write down the transition probability matrix and draw the transition graph of this MC.

2. A standard die is rolled repeatedly. Let [25]

- X_n be the largest value seen in the first n rolls,
- S_n be the number of 6s seen in the first n rolls,
- T_n be the number of 6s seen on rolls $n-1$ and n.

(so $T_n = S_n - S_{n-2}$).
Which of $X_n,\ S_n,\ T_n$ are Markov chains?
For those that are draw their transition graphs.
Hint. Use the definition of a MC. Compare $P\{T_3 = 1|T_2 = 1, T_1 = 1$ and $P\{T_3 = 1|T_2 = 1, T_1 = 0$.

3. Consider a MC X_n on $S = (1,2,3)$ whose transition probability matrix is given by [25]

$$\mathbb{P} = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.4 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

Suppose that the initial distribution $p_i = P\{X_0 = i\} = \frac{1}{3}$ for all $i \in S$. Prove that $P\{X_n = j\} = \frac{1}{3}$ for all n and $j \in S$. Can you deduce a general result from this example?

4. Initially, an urn contains 8 balls of which 4 are red and 4 are green. Two balls are selected from the urn. If one of them is red and the other is green then they are discarded and replaced by two blue balls. In all other cases the selected balls are returned to the urn. This process repeats until the urn contains only blue balls. Let X_n denote the number of red balls in the urn after n draws. Show that X_n is a Markov chain and find its transition matrix. [20]

5. Consider the following simplified model for the spread of disease. The population size is $N = 4$. Some of these people are diseased and all others are healthy. During any single period of time two randomly chosen people interact with each other. Any pair of people has the same chance to be chosen as any other pair. If one of the persons is diseased and the other not, then the disease is transmitted with probability $p = 0.2$. In all other cases no transmission takes place. Let X_n be the number of diseased persons at time n. Specify the transition probability matrix for the MC X_n. [20]