Recurrence

1 Definitions and main statements

Let $X_n,\ n = 0, 1, 2, ...$ be a MC with the state space $S = (1, 2, ...)$, transition probabilities $p_{ij} \stackrel{\text{def}}{=} P\{X_{n+1} = j \mid X_n = i\}$, and the transition matrix $\mathbb{P} = (p_{ij})_{i,j \in S}$.

Definition 1.1
A state $i \in S$ is said to be recurrent if

$$P\{X_n = i \text{ for some } n \geq 1 | X_0 = i\} = 1.$$

In words: $i \in S$ is recurrent if the probability that the MC starting from i will return to i is 1.
A state i is said to be transient if it is non-recurrent. In other words, i is said to be transient if the probability that the Markov chain starting from i will never to return to i is strictly positive.

As usual, we would like to be able to tell whether or not a MC is recurrent in terms of properties of transition probabilities of this chain. Remember that $p_{ij}^{(n)} \stackrel{\text{def}}{=} P\{X_n = j \mid X_0 = i\}$ and that these probabilities can be found, at least in principle, in terms of $\mathbb{P}$, namely $(p_{ij}^{(n)}) = \mathbb{P}^n$.

Theorem 1.2
A state $i \in S$ is recurrent if and only if

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty. \tag{1}$$

Definition 1.3
We say that the MC X_n is recurrent if all states of this MC are recurrent.

Theorem 1.2 will be proved in the next section. But we start with two examples of concrete Markov chains for which this theorem allows one to establish recurrence or transience (whichever is appropriate).

Example 1. Let X_n be a finite regular MC. Then this chain is recurrent. Indeed, since regular chains have the property that $\lim_{n\to\infty} p_{ki}^{(n)} = w_i > 0$, it follows that $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$.
A different and, in a sense, much more natural proof of this statement will be given in Section 3.

Example 2. To define a *simple one-dimensional random walk* imagine a particle moving along a one-dimensional lattice $\mathbb{Z} = \{...-2, -1, 0, 1, 2, ...\}$. The particle is observed at equidistant time moments $n = 0, 1, 2....$ Let X_n be the coordinate of the particle at time n. If at time n the particle is at site k, then at time $n + 1$ it will jump to the right with probability p or to the left with probability $q \stackrel{\text{def}}{=} 1-p$ and the of length of the jump is one; by definition, each next move of the particle does not depend on the history of its movements (or, one could say, on its past positions). It is the clear that the sequence X_n forms a MC with transition probabilities $P\{X_{n+1} = k+1|X_n = k\} = p$, $P\{X_{n+1} = k-1|X_n = k\} = q$. We shall suppose that $X_0 = 0$.

Remark. Note that this MC has an infinite state space - the set of all integer numbers. In the context of the theory of random walks, it is common to call this space state the one-dimensional lattice.

Statement. The state 0 is recurrent if $p = q = \frac{1}{2}$ and is transient if $p \neq q$.
To see this, note first that we can compute the probabilities of return to 0 explicitly. Namely, the first and easy remark is that $p_{00}^{(2n+1)} = 0$ because the random walk has to make an even number of steps in order to return to the starting point. Next, $p_{00}^{(2n)} = \binom{2n}{n} p^n q^n$ (prove this or see the explanation given in the lecture). The Statement will follow if we show that the series

$$\sum_{n=1}^{\infty} p_{ij}^{(n)} \equiv \sum_{n=1}^{\infty} \binom{2n}{n} p^n q^n$$

converges when $p \neq q$ and diverges when $p = q = 0.5$.
The case $p \neq q$ is simple because the ratio test can be applied. Namely, it is easy to check that

$$\lim_{n\to\infty} \frac{\binom{2n+2}{n+1} p^{n+1} q^{n+1}}{\binom{2n}{n} p^n q^n} = \lim_{n\to\infty} \frac{2(2n+1)pq}{n+1} = 4pq < 1 \quad \text{if} \quad p \neq q.$$

(We use here the inequality $p(1-p) < \frac{1}{4}$ if $p \neq 0.5$). Hence the series converges and the random walk is transient.
The ratio test does not work when $p = q = 0.5$ because then $4pq = 1$. We shall prove that in this case the series diverges using the famous Stirling's formula: $n! \sim \sqrt{2\pi n} n^n e^{-n}$. You can check that this formula implies

$$p_{00}^{(2n)} = \binom{2n}{n} p^n q^n = \frac{(2n)!}{(n!)^2} p^n q^n \sim \frac{(4pq)^n}{\sqrt{\pi n}}.$$

When $p = q = 0.5$ we obtain

$$p_{00}^{(2n)} \sim \frac{1}{\sqrt{\pi n}}.$$

The series thus diverges which, according to the above theorem, implies the recurrence of the random walk.

Remark. We say that $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

2 Proof of Theorem 1.2

In order to prove the theorem we need to introduce the following random variable:

$$R_i = \text{the time of the first return to } i. \tag{2}$$

The probability mass function of R_i shall be denoted $f_i^{(n)}$, $n \geq 1$. Obviously,

$$f_i^{(n)} = P\{R_i = n | X_0 = i\} = P\{X_n = i, X_k \neq i \text{ for } k = 1, ..., n-1 | X_0 = i\}. \quad (3)$$

Clearly $f_i^{(1)} = p_{ii}$. All other probabilities $f_i^{(n)}$ can be calculated recursively terms of $p_{ii}^{(k)}$ because of the following

Lemma 2.1
For $n \geq 1$

$$p_{ii}^{(n)} = f_i^{(1)} p_{ii}^{(n-1)} + f_i^{(2)} p_{ii}^{(n-2)} + ... + f_i^{(n-1)} p_{ii}^{(1)} + f_i^{(n)} \equiv \sum_{k=1}^{n} f_i^{(k)} p_{ii}^{(n-k)}. \quad (4)$$

Indeed, if (4) holds then

$$f_i^{(n)} = p_{ii}^{(n)} - (f_i^{(1)} p_{ii}^{(n-1)} + f_i^{(2)} p_{ii}^{(n-2)} + ... + f_i^{(n-1)} p_{ii}^{(1)}) \equiv p_{ii}^{(n)} - \sum_{k=1}^{n-1} f_i^{(k)} p_{ii}^{(n-k)}$$

and hence we can calculate $f_i^{(2)}$ using $f_i^{(1)}$, then we can calculate $f_i^{(3)}$ using $f_i^{(1)}$ and $f_i^{(2)}$, and so on.

Proof of Lemma 2.1. The event $\{X_n = i\}$ can take place only together with one of mutually exclusive events $\{R_i = k\}$, $k = 1, 2, ..., n$. Hence, by the Total Probability Law,

$$\begin{aligned} P\{X_n = i | X_0 = i\} &= \sum_{k=1}^{n} P\{R_i = k | X_0 = i\} P\{X_n = i | R_i = k, X_0 = i\} \\ &= \sum_{k=1}^{n} f_i^{(k)} P\{X_n = i | R_i = k, X_0 = i\}. \end{aligned}$$

But

$$\begin{aligned} P\{X_n = i | R_i = k, X_0 = i\} &= P\{X_n = i | X_k = i, X_l \neq i \text{ when } 1 \leq l \leq k-1, X_0 = i\} \\ &= P\{X_n = i | X_k = i\}\} = p_{ii}^{(n-k)}, \end{aligned}$$

where the last equality follows from the Markov property. The Lemma is proved. $\square$

Next, set $\beta_i \overset{\text{def}}{=} \sum_{i=1}^{\infty} f_i^{(n)}$. Obviously,

$$\beta_i = P\{X_n = i \text{ for at least one } n \geq 1 | X_0 = i\} \equiv P\{ \text{ MC starting at } i \text{ would return to } i\}.$$

We can now say that the state $i \in S$ is recurrent if and only if $\beta_i = 1$ and Theorem 1.2 can be stated as follows:

Theorem 2.2

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty \quad \textit{if and only if} \quad \beta_i = 1 \quad (5)$$

or, equivalently,

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty \quad \textit{if and only if} \quad \beta_i < 1 \tag{6}$$

Proof of Theorem 2.2. Suppose first that $U \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$; we shall now prove that then $\beta_i < 1$. To this end, let us write equations (4) as follows:

$$\begin{aligned} p_{ii}^{(1)} &= f_i^{(1)} \\ p_{ii}^{(2)} &= f_i^{(1)} p_{ii}^{(1)} + f_i^{(2)} \\ p_{ii}^{(3)} &= f_i^{(1)} p_{ii}^{(2)} + f_i^{(2)} p_{ii}^{(1)} + f_i^{(3)} \\ &\vdots \\ p_{ii}^{(n)} &= f_i^{(1)} p_{ii}^{(n-1)} + f_i^{(2)} p_{ii}^{(n-2)} + ... + f_i^{(n-1)} p_{ii}^{(1)} + f_i^{(n)} \end{aligned} \tag{7}$$

Define $U_N \stackrel{\text{def}}{=} \sum_{n=1}^{N} p_{ii}^{(n)}$. Let us add up the first N equations in (7). We obtain

$$U_N = f_i^{(1)}(1 + U_{N-1}) + f_i^{(2)}(1 + U_{N-2}) + ... + f_i^{(N-1)}(1 + U_1) + f_i^{(N)}. \tag{8}$$

Since $\lim_{N\to\infty} U_N = U < \infty$, we can pass to the limit in (8) and obtain

$$U = f_i^{(1)}(1 + U) + f_i^{(2)}(1 + U) + ... + f_i^{(n)}(1 + U) + ... = \beta_i(1 + U). \tag{9}$$

Hence

$$\beta_i = \frac{U}{1 + U} < 1 \tag{10}$$

(which means that the random walk is transient).
Let us now consider the case $U = \infty$ which by definition means that $\lim_{N\to\infty} U_N = \infty$. We shall again make use of (8). Namely, let us replace all the factors of the form $1 + U_n$ in the right hand side of (8) by $1 + U_N$. Since $1 + U_n \leq 1 + U_N$ for all $n \leq N$, we obtain:

$$\begin{aligned} U_N \leq& f_i^{(1)}(1 + U_N) + f_i^{(2)}(1 + U_N) + ... + f_i^{(N-1)}(1 + U_N) + f_i^{(N)}(1 + U_N) \\ =& (f_i^{(1)} + f_i^{(2)} + ... + f_i^{(N)})(1 + U_N) \ \leq \ \beta_i(1 + U_N), \end{aligned} \tag{11}$$

where the last step is due to the fact that $\beta_i = \sum_{n=1}^{\infty} f_i^{(n)}$. Hence

$$\beta_i \geq \frac{U_N}{1 + U_N}$$

and therefore

$$\beta_i \geq \lim_{N\to\infty} \frac{U_N}{1 + U_N} = \lim_{N\to\infty} \frac{1}{\frac{1}{U_N} + 1} = 1.$$

But $\beta_i \leq 1$ because this number is probability of some event. Hence $\beta_i = 1$ which means that the random walk is recurrent.

Theorem 2.2 is proved and thus also Theorem 1.2 is proved. $\square$.

3 Recurrence and communication classes

Let, as in section 2, X_n be a MC with a state space S (S may be an infinite set).

Definition 3.1
We say that two states $i, j \in S$ intercommunicate if there are $k \geq 1$ and $l \geq 1$ such that $p_{ij}^{(k)} > 0$ and $p_{ji}^{(l)} > 0$.
In other words the states i and j intercommunicate if the two probabilities, to reach j starting from i and to reach i starting from j, are strictly positive.

We shall right $i \leftrightarrow j$ instead of saying "i intercommunicates with j". By convention, $i \leftrightarrow i$. Note next that if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$ (explain this statement!).

Definition 3.2
We say that a subset $C \subset S$ forms a communication class if any $i, j \in C$ intercommunicate with each other.

Each state $i \in S$ belongs to exactly one communication class, namely the one formed by all states from S which intercommunicate with i (and therefore also intercommunicate with each other). Hence the state space can be partitioned into (non-intersecting) communication classes. If we call them $C_1,\ C_2, ...$ then

$$S = C_1 \cup\ C_2 \cup ... \cup C_k, \quad \text{with} \quad C_i \cap C_j = \emptyset \text{ if } i \neq j.$$

Remarks. 1. If S is a finite set then of course the number of communication classes is finite. If S is an infinite set the number of communication classes can be infinite too; However, we shall not consider such chains.

2. A communication class may consists of just one state; for instance, this happens if i is an absorbing state or if i cannot be reached from any other state.

3. In the literature communication classes are often called *equivalence classes.* They indeed are equivalence classes with respect to the intercommunication property.

Let us first consider the simplest case, $k = 1$, and thus $S = C_1$. Since all states intercommunicate with each other this simply means that the Markov chain is irreducible. (Remember the definition of an irreducible Markov chain!)

Next, let $k = 2$, that is $S = C_1 \cup C_2$. Then there are two further possibilities: (a) No state from from C_2 can be reached from any state from C_1 and vise versa, no state from C_1 can be reached from any state from C_2. (b) No state from C_2 can be reached from any state from C_1, but some sates from C_1 can be reached from some states from C_2 (formally, there is a third possibility which in fact is the same as the second one; what is it?).

Exercises. 1. Consider Markov chains with transition matrices

$$\begin{pmatrix} 0.3 & 0 & 0.7 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.1 & 0.3 \\ 0.4 & 0 & 0.6 & 0 & 0 \\ 0.5 & 0.1 & 0.2 & 0.2 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0.7 & 0 & 0.3 & 0 & 0 \\ 0 & 0.2 & 0 & 0.5 & 0.3 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.8 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0 & 0.4 & 0.3 \end{pmatrix} \tag{12}$$

What are the communication classes of these chains? Which of the two cases (a) and (b) listed above describe the behavior of each chain?

2. Suppose that $S = C_1 \cup C_2 \cup C_3$. What are the different types of behaviour, in terms of communications between the classes, which one may observe?

We shall now prove a simple but important theorem which, as will be seen, describes the recurrence properties of communications classes.

Theorem 3.3
If i and j are intercommunicating states and i is recurrent then j is recurrent.

Proof. Since i and j intercommunicate, there is a k such that $p_{ij}^{(k)} > 0$ and and an l such that $p_{ji}^{(l)} > 0$. But then

$$p_{jj}^{(k+n+l)} \geq p_{ji}^{(l)} p_{ii}^{(n)} p_{ij}^{(k)}.$$

This inequality holds because one of the possibilities to reach j starting from j in $k+n+l$ transitions would be to first reach i in l transitions (with probability $p_{ji}^{(l)}$), then to return to i after n transitions (with probability $p_{ii}^{(n)}$) and finally to reach j from i in k transitions (with probability $p_{ij}^{(k)}$). Hence

$$\sum_{n=1}^{\infty} p_{jj}^{(n)} \geq \sum_{n=1}^{\infty} p_{jj}^{(k+n+l)} \geq p_{ji}^{(l)} p_{ij}^{(k)} \sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$$

because i is recurrent and this means that $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$. Theorem 3.3 is proved. □

Obviously, Theorem 3.3 implies that both recurrence and transience are properties of a communication class: either all states in the class are recurrent or all of them are transient.

Examples. 1. Consider a finite irreducible Markov chain. It follows from Theorem 3.3 that it is recurrent. Indeed, since S it is a finite set, there must be at least one state which is visited by the chain infinitely many times as $n \to \infty$ and hence is recurrent. But this state intercommunicates with all other states. Hence the chain is recurrent.

In particular, every finite regular Markov chain is recurrent (because regularity implies irreducibility).

In example 1, Section 1, the existence of a more natural prof of recurrence was mentioned. So, here it is.

2. Consider the one-dimensional random walk discussed in Section 1. Since 0 intercommunicates with each state of the lattice, we conclude that all states are recurrent if $p = q = \frac{1}{2}$ and are transient otherwise.

Exercise. For each of the two Marcov chains whose transition matrices are given by (12) establish which states are recurrent and which are transient?