# Probability III – 2007/08

## Solutions to Exercise Sheet 7

1. In the fast mode we have an $M(1)/M(2)/1$ queue. In the slow mode we have an $M(1)/M(\frac{6}{5})/1$ queue. We know from lectures that the number of customers in an $M(\lambda)/M(\mu)/1$ queue in the long run has distribution $\text{Geom}(1-\rho)$ where $\rho = \frac{\lambda}{\mu}$. So the mean queue length is approximately $\frac{\rho}{1-\rho}$.

For the fast mode the mean queue length is therefore close to $\frac{1/2}{1-1/2} = 1$. Whereas, for the slow mode the mean queue length is approximately $\frac{5/6}{1-5/6} = 5$.

2. Let $Q(t)$ be the length of the queue at time $t$. From lectures we know that there is a limiting distribution. That is

$$\mathbb{P}(Q(t) = i) \to w_i,$$

where

$$w_0 = \frac{1-\rho}{1+\rho},$$

and

$$w_n = 2\rho^n \frac{1-\rho}{1+\rho}$$

for all $n \geq 1$. Armed with this we can answer the question. Noting that $\rho = \frac{1}{4}$ for our queue.

(You should make sure that you can derive this limiting distribution although I am happy for you to quote it if you are not specifically asked for a derivation.)

i) This is just

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}(Q(t) = 0) &= w_0 \\ &= \frac{3}{5}.\end{aligned}$$

ii) This is clearly the complimentary event to part i) and so its probability is $1 - \frac{3}{5} = \frac{2}{5}$.

iii) Here we want

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}(Q(t) \geq 2) &= 1 - w_0 - w_1 \\ &= 1 - \frac{3}{5} - \frac{3}{10} \\ &= \frac{1}{10}.\end{aligned}$$

iv) There are 2 customers waiting to be served at time $t$ if and only if $Q(t) = 4$ (2 are being served and 2 are waiting to be served). Thus, we want

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}(Q(t) = 4) &= w_4 \\ &= 2\frac{1}{4^4}\frac{3}{5} \\ &= \frac{3}{640}. \end{aligned}$$

3. This is very similar to finding the equlibrium distribution for the $M(\lambda)/M(\mu)/2$ queueing system.

First, note that the $M(\lambda)/M(\mu)/3$ queueing system is a birth-death process with parameters $\lambda_k = \lambda$ for all $k \geq 0$, $\mu_0 = 0$, $\mu_1 = \mu$, $\mu_2 = 2\mu$, $\mu_k = 3\mu$ for all $k \geq 3$. As before, we know that if this birth-death process has an equilibrium distribution then it must satisfy

$$w_i = \frac{\lambda_0\lambda_1\ldots\lambda_{i-1}}{\mu_1\mu_2\ldots\mu_i} w_0.$$

Hence we must have

$$\begin{aligned} w_1 &= \frac{\lambda}{\mu} w_0, \\ w_2 &= \frac{\lambda^2}{2\mu^2} w_0, \\ w_3 &= \frac{\lambda^3}{6\mu^3} w_0, \\ w_4 &= \frac{\lambda^4}{18\mu^4} w_0, \end{aligned}$$

and so on. (Make sure that you can see why this follows from result on the equilibrium distribution of the birth-death process).

More generally,

$$\begin{aligned} w_1 &= \frac{\lambda}{\mu} w_0, \\ w_n &= \left(\frac{\lambda}{3\mu}\right)^n \frac{9}{2} w_0, \end{aligned}$$

for all $n \geq 2$.

For this to be an equilibrium distribution we must also have $\sum_{i\geq 0} w_i = 1$. That is

$$w_0 \left(1 + \frac{\lambda}{\mu} + \frac{9}{2}\sum_{i\geq 2}\left(\frac{\lambda}{3\mu}\right)^i\right) = 1.$$

We can do this if $\frac{\lambda}{3\mu} < 1$. In this case, letting $\rho = \frac{\lambda}{3\mu}$, we must take

$$\begin{aligned} w_0 &= \left(1 + 3\rho + \frac{9}{2}\frac{\rho^2}{1-\rho}\right)^{-1} \\ &= \frac{1-\rho}{1-\rho+3\rho-3\rho^2+\frac{9}{2}\rho^2} \\ &= \frac{2-2\rho}{3\rho^2+4\rho+2}. \end{aligned}$$

We deduce that the equilibrium distribution is

$$w_i = \begin{cases} \frac{2-2\rho}{3\rho^2+4\rho+2} & \text{if } i = 0. \\ 3\rho\frac{2-2\rho}{3\rho^2+4\rho+2} & \text{if } i = 1. \\ \frac{9}{2}\rho^i\frac{2-2\rho}{3\rho^2+4\rho+2} & \text{if } i \geq 2. \end{cases}$$

4. Since the service times are all exactly $d$, we have that $Q(t)$ is just the number of customers who arrived in the interval $(t-d,t]$. Since the arrivals form a Poisson process this is distributed $\text{Po}(d\lambda)$. Simple as that.

5. Let $S_1, S_2, \ldots, S_n$ denote the service times of the $n$ customers who are in the system when the customer in question arrives. The arriving customer has to wait for time $T = S_1 + S_2 + \cdots + S_n$ to be served (you might have interpreted the question as asking for the total time waiting in the system in which case you'd have an $S_{n+1}$ term representing the new customers service time – sorry for the ambiguity). We know that the $S_i$ are independent and each is distributed $\text{Exp}(\mu)$, and so

$$\begin{aligned} \mathbb{E}(T) &= n\mathbb{E}(S_1) \\ &= \frac{n}{\mu}. \end{aligned}$$

I should have said for the second part that we assume that $\lambda < \mu$ so that the equilibrium distribution exists.

Let $T_E$ the time a new customer waits at equilibrium. We use the fact that

$$\begin{aligned} \mathbb{E}(T_E) &= \sum_{n\geq 0} \mathbb{E}(\text{time to wait}|Q(t) = n)\mathbb{P}(Q(t) = n) \\ &= \sum_{n\geq 0} \frac{n}{\mu} w_n \end{aligned}$$

where $w_n$ is the equilibrium distribution, and we have used the first part of the question for the conditional expectation. Hence

$$\mathbb{E}(T_E) = \frac{1}{\mu}\sum_{n\geq 0} n(1-\rho)\rho^n.$$

The sum is the expectation of a $\text{Geom}(1-\rho)$ random variable and so,

$$\mathbb{E}(T_E) = \frac{1}{\mu}\frac{1}{1-\rho} = \frac{1}{\mu-\lambda}.$$

If you had assumed the question asked for expectation of the total time waiting you would obviously have got $\frac{1}{\mu-\lambda}+\frac{1}{\mu}$.

**Please let me know if you have any comments or corrections**