Probability III – 2007/08

Solutions to Exercise Sheet 6

1. For the first two parts all we need is that if k births have happened then the time until the next birth is as it would be in a Poisson process of rate $k\lambda$.

i) $\mathbb{P}(\text{no births in } (0,4]) = e^{-4\lambda_0} = e^{-12}$.

ii) $\mathbb{P}(\text{no births in } (5,6] | X(5) = 3) = e^{-\lambda_3} = e^{-15}$.

iii) As for the Poisson process we have

$$W_i = S_0 + S_1 + \cdots + S_{i-1}.$$

Each S_i is distributed exponentially with parameter λ_i, and so,

$$\begin{aligned}\mathbb{E}(W_4) &= \mathbb{E}(S_0) + \mathbb{E}(S_1) + \mathbb{E}(S_2) + \mathbb{E}(S_3) \\ &= \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{15} \\ &= \frac{32}{45}.\end{aligned}$$

2.

i) Essentially, the number of sales is the sum of independent Poisson processes. If $X(t) = k$ there are k purchasers each of which generates another purchaser according to a Poisson process of rate β The media advertising which generates another purchaser occording to a Poisson process of rate α. It follows that

$$\mathbb{P}(X(t+h) = k+l | X(t) = k) = \begin{cases} \alpha h + k\beta h + o(h) & \text{if } l = 1 \\ 1 - \alpha h + k\beta h + o(h) & \text{if } l = 0 \\ o(h) & \text{if } l > 1 \\ 0 & \text{if } l < 0 \end{cases}$$

These are all obvious apart from the $l = 0$ case where the probability in question is

$$(1 - \alpha h)(1 - \beta h)^k = 1 - \alpha h - k\beta h + o(h)$$

as claimed.

The rest of the axioms for a birth process are easiy checked. Hence, we do have a birth process with parameters $\lambda_k = \alpha + k\beta$.

(Compare this with the linear birth process.)

ii) In lectures we derived the following differential equations:

$$\begin{aligned} p_0'(t) &= -\lambda_0 p_0(t) \\ p_1'(t) &= \lambda_0 p_0(t) - \lambda_1 p_1(t) \\ p_2'(t) &= \lambda_1 p_1(t) - \lambda_2 p_2(t). \end{aligned}$$

(You could also think of these as a special case of the forwards equations, see question 5, with $i = 0$ and $\mu_k = 0$.)

iii) In lectures we showed that these differential equations always have a unique the solution. The method of that proof can be used to solve them.

Firstly, it is clear that the first equation has solution

$$p_0(t) = Ce^{-t}.$$

Further, we know that $p_0(0) = 1$ and so $C = 1$. Hence,

$$p_0(t) = e^{-t}.$$

Rearranging the second equation gives

$$\frac{d}{dt}\left(p_1(t)e^{\lambda_1 t}\right) = e^{\lambda_1 t}\lambda_0 p_0(t) = e^{2t}.$$

And so, integrating and using the boundary condition $p_1(0) = 0$,

$$p_1(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}.$$

Similarly, rearranging the third equation

$$\frac{d}{dt}\left(p_2(t)e^{\lambda_2 t}\right) = e^{\lambda_2 t}\lambda_1 p_1(t) = \frac{3}{2}e^{4t} - \frac{3}{2}e^{2t}.$$

And so, integrating and using the boundary condition $p_2(0) = 0$,

$$p_2(t) = \frac{3}{8}e^{-t} - \frac{3}{4}e^{-3t} + \frac{3}{8}e^{-5t}.$$

The required probability is $p_2(3)$ which is

$$\frac{3}{8}e^{-3} - \frac{3}{4}e^{-9} + \frac{3}{8}e^{-15}.$$

3. The relevant result here is that the process explodes if and only if $\sum_{k\geq 0} \lambda_k^{-1}$ is finite. This is easily checked in each case.

a) $\sum_{k\geq 0} 2^{-k} = 2$ so the process explodes.

b) $\sum_{k\geq 0} \frac{1}{9}$ is infinte so the process does not explode.

c) I should really have said $\lambda_k = 7k$ for $k \geq 1$, $\lambda_0 = 1$ since if $\lambda_0 = 0$ the process is rather boring! We get $1 + \sum_{k\geq 1} \frac{1}{7k}$ which is infinite and so the process does not explode.

4.

i) As usual, writing S_i for the time spent with i individuals still alive, we have $T = S_4 + S_3 + S_2 + S_1$. The random variable S_i is exponentially distributed with parameter μ_i. It follows that

$$\begin{aligned}\mathbb{E}(T) &= \mathbb{E}(S_4) + \mathbb{E}(S_3) + \mathbb{E}(S_2) + \mathbb{E}(S_1) \\ &= \frac{1}{6} + \frac{1}{4} + \frac{1}{2} + \frac{1}{2} \\ &= \frac{17}{12}\end{aligned}$$

ii) The S_i defined above are independent. Hence,

$$\begin{aligned}\mathrm{Var}(T) &= \mathrm{Var}(S_4) + \mathrm{Var}(S_3) + \mathrm{Var}(S_2) + \mathrm{Var}(S_1) \\ &= \frac{1}{6^2} + \frac{1}{4^2} + \frac{1}{2^2} + \frac{1}{2^2} \\ &= \frac{85}{144}.\end{aligned}$$

5. The first set of equations are the *backwards equations* for a birth-death process.

$$\begin{aligned}p_{i,j}(t+h) &= \sum_{k\geq 0} p_{i,k}(h) p_{k,j}(t) \qquad \text{(Chapman-Kolmogorov)} \\ &= \mu_i h p_{i-1,j}(t) + (1 - \mu_i h - \lambda_i h) p_{i,j}(t) + \lambda_i h p_{i+1,j}(t) + o(h)\end{aligned}$$

This is because all other terms in the sum are $o(h) p_{k,j}(t)$. Dividing by h,

$$\frac{p_{i,j}(t) - p_{i,j}(t+h)}{h} = \mu_i p_{i-1,j}(t) - (\mu_i + \lambda_i) p_{i,j}(t) + \lambda_i p_{i+1,j}(t) + \frac{o(h)}{h}.$$

Now, letting h tend to 0 we obtain the desired equation,

$$p'_{i,j}(t) = \mu_i p_{i-1,j}(t) - (\lambda_i + \mu_i) p_{i,j}(t) + \lambda_i p_{i+1,j}(t).$$

The *forwards equations* are derived in a similar way, starting from a different application of the Chapman-Kolmogorov relation. Specifically,

$$\begin{aligned} p_{i,j}(t+h) &= \sum_{k\geq 0} p_{i,k}(t)p_{k,j}(h) \quad \text{(Chapman-Kolmogorov)} \\ &= \lambda_{j-1}hp_{i,j-1}(t) + (1-\mu_j h - \lambda_j h)p_{i,j}(t) + \mu_{j+1}hp_{i,j+1}(t) + o(h). \end{aligned}$$

And so,

$$\frac{p_{i,j}(t) - p_{i,j}(t+h)}{h} = \lambda_{j-1}p_{i,j-1}(t) - (\mu_j + \lambda_j)p_{i,j}(t) + \mu_{j+1}p_{i,j+1}(t) + \frac{o(h)}{h}.$$

Letting h tend to 0 we obtain,

$$p'_{i,j}(t) = \lambda_{j-1}p_{i,j-1}(t) - (\lambda_j + \mu_j)p_{i,j}(t) + \mu_{j+1}p_{i,j+1}(t).$$

Please let me know if you have any comments or corrections