Probability III – 2007/08

Please hand in solutions by 12:00, December 6

Exercise Sheet 6

1. Let $X(t)$ be a birth process with $X(0) = 0$ and intensities $\lambda_k = k^2 + k + 3$. Calculate the following:

i) the probability that there are no births in the interval $(0,4]$.

ii) the probability that there are no births in the interval $(5,6]$ conditioned on $X(5) = 3$.

iii) the expectation of W_4, the time of the fourth birth.

2. A new product is promoted by media advertising and word-of-mouth advertising (where satisfied customers tell others about the product). Suppose that media advertising creates purchasers according to a Poisson process of rate α per month. Suppose further that each purchaser generates sales by word-of-mouth advertising at a rate of β per month. Let $X(t)$ be the total number of sales of the product t months after it is launched. Let

$$p_n(t) = \mathbb{P}(X(t) = n).$$

i) Explain why $X(t)$ is a birth process with with paramaters $\lambda_k = \alpha + k\beta$.

ii) Write down equations for $p_0'(t), p_1'(t)$ and $p_2'(t)$ in terms of $p_0(t), p_1(t)$ and $p_2(t)$..

iii) Suppose that $\alpha = 1$ and $\beta = 2$. Solve the equations for part b). Hence find the probability that 2 purchases are made in the first 3 months.

3. Let $X(t)$ be a birth process with parameters λ_k. In which of the following cases does the process explode:

a) $\lambda_k = 2^k$

b) $\lambda_k = 9$

c) $\lambda_k = 7k$ for $k \geq 1$, $\lambda_0 = 1$

4. A death process with $X(0) = 4$ has parameters, $\mu_0 = 0$, $\mu_1 = 2$, $\mu_2 = 2$, $\mu_3 = 4$, $\mu_4 = 6$. Let T be the time the process takes to reach state 0.

i) Find $\mathbb{E}(T)$.

ii) Find $\mathrm{Var}(T)$.

5. Let $X(t)$ be a birth-death process with parameters λ_k and μ_k. Let

$$p_{i,j}(t) = \mathbb{P}(X(t+s) = j | X(s) = i).$$

Derive the equations:

$$p'_{i,j}(t) = \mu_i p_{i-1,j}(t) - (\lambda_i + \mu_i) p_{i,j}(t) + \lambda_i p_{i+1,j}(t).$$

Derive the equations:

$$p'_{i,j}(t) = \lambda_{j-1} p_{i,j-1}(t) - (\lambda_j + \mu_j) p_{i,j}(t) + \mu_{j+1} p_{i,j+1}(t).$$

(A hint on how to do this was given in lectures.)