Probability III – 2007/08

Solutions to Exercise Sheet 5

1.

a) We know that if $0 < u < t$ then $X(u)$ conditioned on $X(t) = n$ has distribution $\text{Bin}(n, \frac{u}{t})$. Thus,

$$\begin{aligned}\mathbb{P}(X(\frac{1}{2}) = 2|X(1) = 5) &= \binom{5}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^3 \\ &= \frac{5}{16}\end{aligned}$$

b) We have that $S_i = W_i - W_{i-1}$ if $i > 1$, and $S_1 = W_!$. It follows that,

$$W_n = S_n + S_{n-1} + \cdots + S_1.$$

By linearity of expectation and the fact that the S_i are distributed $\text{Exp}(1)$ we have that,

$$\begin{aligned}\mathbb{E}(W_n) &= \mathbb{E}(S_n + \cdots + S_1) \\ &= \mathbb{E}(S_n) + \cdots + \mathbb{E}(S_1) \\ &= n \times 1 \\ &= n\end{aligned}$$

c) The number of particles which exist at time 1 minute is precisely the number of particles which were emitted in the interval $(\frac{5}{6}, 1]$. We know that this number has a $\text{Po}(\frac{1}{6})$ distribution. Thus,

$$\mathbb{P}(k \text{ particles exist at time } 1) = e^{-\frac{1}{6}}\frac{1}{6^k k!}.$$

2. This looks obvious but you do need to be careful that you are justifying every step using the definition on the Poisson process.

Firstly,

$$
\begin{aligned}
\mathbb{P}(X(u+t)=j|X(u)=i) &= \frac{\mathbb{P}(X(u+t)=j,X(u)=i)}{\mathbb{P}(X(u)=i)} \\
&= \frac{\mathbb{P}(X(u+t)-X(u)=j-i,X(u)-X(0)=i)}{\mathbb{P}(X(u)-X(0)=i)} \\
&= \frac{\mathbb{P}(X(u+t)-X(u)=j-i)\mathbb{P}(X(u)-X(0)=i)}{\mathbb{P}(X(u)-X(0)=i)} \\
&= \mathbb{P}(X(u+t)-X(u)=j-i).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathbb{P}(X(u+t)=j|X(u)=i,X(u_1)=i_1) \\
&= \frac{\mathbb{P}(X(u+t)=j,X(u)=i,X(u_1)=i_1)}{\mathbb{P}(X(u)=i,X(u_1)=i_1)} \\
&= \frac{\mathbb{P}(X(u+t)-X(u)=j-i,X(u)-X(u_1)=i-i_1,X(u_1)-X(0)=i_1)}{\mathbb{P}(X(u)-X(u_1)=i-i_1,X(u_1)-X(0)=i_1)} \\
&= \frac{\mathbb{P}(X(u+t)-X(u)=j-i)\mathbb{P}(X(u)-X(u_1)=i-i_1)\mathbb{P}(X(u_1)-X(0)=i_1)}{\mathbb{P}(X(u)-X(u_1)=i-i_1)\mathbb{P}(X(u_1)-X(0)=i_1)} \\
&= \mathbb{P}(X(u+t)-X(u)=j-i).
\end{aligned}
$$

Hence, both sides of the given equation are equal.

The second part is exactly the same except the notation becomes more unpleasant.

3. As usual with continuous random variables, the cdf is easier to work out than the pdf.

$$
\begin{aligned}
F_{W_1|X(t)=n}(u) &= \mathbb{P}(W_1 \le u|X(t)=n) \\
&= 1-\mathbb{P}(W_1 > u|X(t)=n) \\
&= 1-\mathbb{P}(X(u)=0|X(t=n) \\
&= 1-\left(1-\frac{u}{t}\right)^n \quad (\text{since } X(u)|X(t)=n \text{ is Bin}(n,\frac{u}{t}))
\end{aligned}
$$

for $0 < u \le t$.

Differentiating this with respect to u gives the pdf.

$$
f_{W_1|X(t)=n}(u) = \frac{n}{t}\left(1-\frac{u}{t}\right)^{n-1}
$$

for $0 < u \leq t$.

I didn't ask what happened for other values of u, but it is clear that the pdf is 0 outside this range as W_1 certainly lies somewhere in the interval $(0,t]$.

The expectation can be found in the usual way

$$\begin{aligned}
\mathbb{E}(W_1|X(t)=n) &= \int_0^t x\frac{n}{t}\left(1-\frac{x}{t}\right)^{n-1} dx \\
&= \left[-x\left(1-\frac{x}{t}\right)^n\right]_{x=0}^{x=t} + \int_0^t \left(1-\frac{x}{t}\right)^n dx \quad \text{(integrating by parts)} \\
&= 0 - \left[\frac{t}{n+1}\left(1-\frac{x}{t}\right)^{n+1}\right]_{t=0}^{x=t} \\
&= \frac{t}{n+1}.
\end{aligned}$$

4. These are most easily done by considering random variables U_i for $1 \leq i \leq n$. Where the U_i are independent and each is distributed uniformly on $[0,3]$. We know from lectures that if R is a symmetric function of the W_i (as all the functions given are) then

$$\mathbb{E}(R(W_1,\ldots,W_5)|X(3)=5) = \mathbb{E}(R(U_1,\ldots,U_5)).$$

i)

$$\begin{aligned}
\mathbb{E}(W_1+W_2+W_3+W_4+W_5|X(3)=5) &= \mathbb{E}(U_1+U_2+U_3+U_4+U_5) \\
&= 5 \times \frac{3}{2} \\
&= \frac{15}{2}.
\end{aligned}$$

ii)

$$\mathbb{E}(W_1W_2W_3W_4W_5|X(3)=5) = \mathbb{E}(U_1U_2U_3U_4U_5)$$
$$= \mathbb{E}(U_1)\mathbb{E}(U_2)\mathbb{E}(U_3)\mathbb{E}(U_4)\mathbb{E}(U_5) \quad \text{(since the } U_i \text{ are independent)} = \left(\frac{3}{2}\right)^5$$

iii)

$$\mathbb{E}\left(\sum_{i\neq j} W_iW_j|X(3)=5\right) = \mathbb{E}\left(\sum_{i\neq j} U_iU_j\right)$$

$$= 20\mathbb{E}(U_1U_2) \quad \text{(since there are 20 equal terms in the sum)}$$

$$= 20\left(\frac{3}{2}\right)^2 \quad \text{(since the } U_i \text{ are independent)} = 45.$$

5. Let $X(t)$ be the number of faults in the first t miles of cable. Let C be the total cost of repairing all faults in the first M miles of cable. We first condition on $X(M)$ to get

$$\mathbb{E}(C) = \sum_{n\geq 0} \mathbb{E}(C|X(M)=n)\mathbb{P}(X(M)=n).$$

Now, writing W_i for the position of the ith fault (that is the ith waiting time), we have that

$$\mathbb{E}(C|X(M)=n) = \sum_{i=1}^{n}(W_1^k + W_2^k + \cdots + W_n^k).$$

We can use the same trick as for question 4 to get,

$$\mathbb{E}(C|X(M)=n) = \sum_{i=1}^{n}(U_1^k + U_2^k + \cdots + U_n^k),$$

where the U_i are independent random variables, each uniformly distributed on $[0,M]$. Thus,

$$\mathbb{E}(C|X(M)=n) = n\int_0^M \frac{x^k}{M}dx = \frac{nM^k}{k+1}.$$

Putting this into the first equation we have that,

$$\mathbb{E}(C) = \sum_{n\geq 0}\frac{nM^k}{k+1}\mathbb{P}(X(M)=n) = \frac{M^k}{k+1}\sum_{n\geq 0} ne^{-M\lambda}\frac{(M\lambda)^n}{n!} = \frac{M^k}{k+1}\times(M\lambda)$$

The last equality follows because the sum is the expectation of a Po$(M\lambda)$ random variable. We conclude that

$$\mathbb{E}(C) = \frac{M^{k+1}\lambda}{k+1}.$$

Please let me know if you have any comments or corrections