Probability III – 2007/08

Please hand in solutions by 12:00, November 29

Exercise Sheet 5

1. Radioactive particles are emitted from a source according to a Poisson process of intensity 1 per minute.

a) Suppose that 5 particles are emitted in the first minute. What is the probability that exactly 2 of them were emitted in the first 30 seconds.

b) Let W_n be the time of emission of the nth particle. Express W_n in terms of the sojourn times of the process and hence find $\mathbb{E}(W_n)$.

c) Suppose that each particle survives for 10 seconds. What is the probability that k particles exists at time 1 minute.

2. Prove that if $X(t)$ is a Poisson process then

$$\mathbb{P}(X(u+t)=j|X(u)=i,X(u_1)=i_1)=\mathbb{P}(X(u+t)=j|X(u)=i),$$

for all $0<u_1<u$, $t>0$, $i_1\leq i\leq j$.
Thus the Poisson process satisfies a form of Markov property. If you're feeling keen you could try to prove the stronger statement.

$$\mathbb{P}(X(u+t)=j|X(u)=i,X(u_1)=i_1,\dots,X(u_k)=i_k)=\mathbb{P}(X(u+t)=j|X(u)=i),$$

for all $0<u_k<\cdots<u_1<u$, $t>0$, $i_k\leq\cdots\leq i_1\leq i\leq j$.

3. Show that the probability density function of W_1 conditioned on $X(t)=n$ for $n>0$ is

$$f_{W_1|X(t)=n}(u)=\frac{n}{t}\left(1-\frac{u}{t}\right)^{n-1}.$$

for $0<u\leq t$.
What is $\mathbb{E}(W_1|X(t)=n)$?

4. Let $X(t)$ be a Poisson process of rate 2. Let $W_1,W_2,\dots$ be the waiting times of the process. Calculate the following:

i) $\mathbb{E}(W_1+W_2+W_3+W_4+W_5|X(3)=5)$

ii) $\mathbb{E}(W_1W_2W_3W_4W_5|X(3)=5)$

iii) $\mathbb{E}(\sum_{i\neq j}W_iW_j|X(3)=5)$

5. A long cable has faults on it which form a Poisson process of intensity λ per mile. The cost of repairing a fault at position u on the cable is £u^k for some constant k. What is the expected cost of repairing all faults in the first M miles of cable?