Probability III – 2007/08

Solutions to Exercise Sheet 4

1. Let the state space for the chain be $\{1,2,3,4,5,6\}$. From the transition matrix it can be seen that the equivalence classes are:

$$\{1\},\{2,5\},\{3,4\},\{6\}.$$

If you're having trouble seeing this then try drawing the tranistion graph.

States 1 and 6 are absorbing so are certainly recurrent. States 2 and 5 are also recurrent (the probability of not returning to state 2 by n steps is $\frac{1}{2}^n \to 0$). Finally, states 3 and 4 are transient (the probability that we leave $\{3,4\}$ eventually is 1 and having left we can never return).

2. The probability that of the first n steps, k of them were to the right is $2^{-n}\binom{n}{k}$. If we return to 0 after n steps then we must have made exactly twice as many to the left as to the right. Thus

$$p_{00}^{(n)} = \begin{cases} 2^{-n}\binom{n}{\frac{n}{3}} & \text{if } n \text{ is a multiple of 3} \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$\begin{aligned}
\sum_{k\geq 0} p_{00}^{(k)} &= \sum_{k\geq 0} 2^{-3k}\binom{3k}{k} \\
&= \sum_{k\geq 0} 2^{-3k}\frac{(3k)!}{k!(2k)!} \\
&\leq \sum_{k\geq 0} 2^{-3k} C \frac{(3k)^{(3k)}\sqrt{3k}e^k e^{2k}}{e^{3k}k^k\sqrt{k}(2k)^{2k}\sqrt{2k}} \qquad \text{(for some constant } C\text{)} \\
&= \sum_{k\geq 0} C \frac{3^{3k}\sqrt{3}}{2^{3k}2^{2k}\sqrt{k}} \\
&\leq \sqrt{3}\sum_{k\geq 0}\frac{27^k}{32^k} \\
&= \frac{5\sqrt{3}}{32} \qquad \text{(sum of a geometric progression)}
\end{aligned}$$

We deduce that since $\sum_{k\geq 0} p_{00}^{(k)}$ is finite, state 0 is transient. Since the chain is irreducible this means that every state is transient.

3. Using the notation of the lectures we have that

$$p_k = (1-p)p^k,$$

and hence,

$$\begin{aligned} g(z) &= \sum_{k\geq 0} p_k z^k \\ &= \frac{1-p}{1-pz}. \end{aligned}$$

You might have noticed that the p_k have geometric distribution, so $g(z)$ is the generating function for a geometric random variable.
We know that u, the probability of eventual extinction, is 1 if $g'(1) \leq 1$. Differentiating g gives,

$$g'(z) = \frac{p(1-p)}{(1-pz)^2}.$$

Hence, $g'(1) = \frac{p}{1-p}$. This is ≤ 1 if $p \leq \frac{1}{2}$.
If $p < \frac{1}{2}$ then $g'(1) > 1$ and so we know from the general theory that $g(z) = z$ has two roots, 1 and a smaller root in the interval $(0,1)$. We also know that u is equal to the smaller root. So to find u we need to solve

$$u = \frac{1-p}{1-pu}.$$

That is,

$$\begin{aligned} pu^2 - u + 1 - p &= 0 \\ (pu-(1-p))(u-1) &= 0 \end{aligned}$$

Hence $u = \frac{1-p}{p}$. So if $p < \frac{1}{2}$ then the probability of eventual extinction is $\frac{1-p}{p}$.
By the general theory developed in lectures weknow that $G_3(z)$, the generating function for X_3, satisfies

$$G_3(z) = g(g(g(z))).$$

Whenn $p = \frac{1}{3}$, we have that $g(z) = \frac{2}{3-z}$. Thus,

$$\begin{aligned} G_3(z) &= g(g\left(\frac{2}{3-z}\right) \\ &= g\left(\frac{2}{3-\frac{2}{3-z}}\right) \\ &= g\left(\frac{6-2z}{7-3z}\right) \\ &= \frac{2}{3-\frac{6-2z}{7-3z}} \\ &= \frac{14-6z}{15-7z}. \end{aligned}$$

4. When reading these solutions, make sure that you can see which properties of the Poisson process we are using to justify each step below.

a)

$$\begin{aligned}\mathbb{P}(X(4)=2) &= \mathbb{P}(X(4)-X(0)=2)\\ &= e^{-4\times 4}\frac{(4\times 4)^2}{2!}\\ &= 128e^{-16}.\end{aligned}$$

b)

$$\begin{aligned}\mathbb{P}(X(4)=2, X(1)=1) &= \mathbb{P}(X(1)-X(0)=1)\mathbb{P}(X(4)-X(1)=1)\\ &= e^{-4}4e^{-12}12\\ &= 48e^{-16}.\end{aligned}$$

c)

$$\begin{aligned}\mathbb{P}(X(4)=2|X(1)=1) &= \mathbb{P}(X(4)-X(1)=1)\\ &= 12e^{-12}.\end{aligned}$$

d)

$$\begin{aligned}\mathbb{P}(X(1)=1|X(4)=2) &= \frac{\mathbb{P}(X(1)=1, X(4)=2)}{\mathbb{P}(X(4)=2)}\\ &= \frac{48e^{-16}}{128e^{-16}}\\ &= \frac{3}{8}.\end{aligned}$$

e)

$$\mathbb{P}(X(4)=1|X(1)=2)=0.$$

5. Let $X(t)$ be the number of cracks in the first t miles and $Y(t)$ be the number of punctures in the first t miles. We must check that $Z(t) = X(t) + Y(t)$ is a Poisson process.

Firstly, it is clear that $Z(0) = 0$ (since the fact that X and Y are Poisson processes means that $X(0) = Y(0) = 0$). Secondly, we have that for any $t_1 < t_2 < \cdots < t_n$, $X(t_{i+1}) - X(t_i)$, $Y(t_{i+1}) - Y(t_i)$ are independent random variables. Hence, $Z(t_{i+1}) - Z(t_i)$ are independent random variables. Finally, we must show that $Z(t+s) - Z(s)$ has a $\mathrm{Po}(t(\lambda+\mu))$ distribution. Now,

$$Z(t+s) - Z(s) = X(t+s) - X(s) + Y(t+s) - Y(s).$$

We know that $X(t+s) - X(s)$ is distributed $\mathrm{Po}(t\lambda)$ and $Y(t+s) - Y(s)$ is distributed $\mathrm{Po}(t\mu)$. Moreover, these two random variables are independent. It follows (by standard properties of the Poisson distribution – see for example exercise sheet 1 question 1 (b)) that their sum is distributed $\mathrm{Po}(t(\lambda+\mu))$, as required.

Let $F(t)$ be the number of un-repairable faults ni the first t miles. Again, it is immediate that $F(0) = 0$, and that for any $t_1 < t_2 < \cdots < t_n$, $F(t_{i+1}) - F(t_i)$ are independent random variables. By standard properties of the Poisson distribution (see for example exercise sheet 1 question 3) we have that $F(t+s) - F(s)$ is distributed $\mathrm{Po}(p\lambda)$ as required.

Please let me know if you have any comments or corrections