Probability III – 2007/08

Solutions to Exercise Sheet 3

1. Let X_n denote the number of red balls in the urn after n time steps. The process follows the Markov chain on state space $S = \{0,1,2,3,4,5\}$, with $X_0 = 5$ and transition probabilities

$$p_{ij} = \begin{cases} \frac{3}{3+i} & \text{if } j = i \\ \frac{i}{3+i} & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

(Try drawing the transition graph if you can't see what's going on.)

State 0 is the only absorbing state and the chain is finite so eventually it must be absorbed. Now let

$$w_i = \mathbb{E}(\text{time to absorbtion} | X_0 = i).$$

By the method of first step analysis:

$$\begin{aligned} w_1 &= 1 + \frac{3}{4} w_1 \\ w_2 &= 1 + \frac{2}{5} w_1 + \frac{3}{5} w_2 \\ w_3 &= 1 + \frac{1}{2} w_2 + \frac{1}{2} w_3 \\ w_4 &= 1 + \frac{4}{7} w_3 + \frac{3}{7} w_4 \\ w_5 &= 1 + \frac{5}{8} w_4 + \frac{3}{8} w_5. \end{aligned}$$

The first equation gives that $w_1 = 4$. This can be substituted into the second equation to find w_2, which can be substitued into the third equation and so on. The result is

$$\begin{aligned} w_1 &= 4 \\ w_2 &= \frac{13}{2} \\ w_3 &= 1 + \frac{17}{2} \\ w_4 &= \frac{41}{4} \\ w_5 &= \frac{237}{20}. \end{aligned}$$

So the expected duration of the process is $\frac{237}{20}$ time steps.

2.

a) Every state can be reached from every other in a finite number of steps. Also, $p_{11} = 1/10 > 0$ and so (by the sufficient condition for regularity given in lectures) the chain is regular.

To find the equilibrium distribution we need to solve:

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} \begin{pmatrix} 1/10 & 1/2 & 0 & 2/5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix}.$$

These have solution

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} = \begin{pmatrix} 10/29 & 5/29 & 5/29 & 9/29 \end{pmatrix}.$$

We deduce that in the long run the proportion of time spent in state i will be close to w_i, for these values of w_i.

b) There was a mistake in the original version of the sheet. Hopefully everyone got the corrected version.

You can check for any $i, j \in S$ there is a positive probability of going from i to j in 6 steps. To do this either calculate P^6 and notice that all entries are positive, or look for paths of length 6 between every pair of states in the transition graph. This means that the chain is regular.

To find the equilibrium distribution we need to solve:

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix}.$$

These have solution

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} = \begin{pmatrix} 3/7 & 1/7 & 1/7 & 2/7 \end{pmatrix}.$$

We deduce that in the long run the proportion of time spent in state i will be close to w_i, for these values of w_i.

3.

a) The chain is irreducible but not regular (since $p_{11}^{(k)} = 0$ if k is odd, and $p_{12}^{(k)} = 0$ if k is even). The method used in the previous question shows that there is a unique equilibrium distribution,

$$\begin{pmatrix} w_1 & w_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}.$$

b) The chain is irreducible but not regular (since $p_{11}^{(k)} = 0$ if k is odd, and $p_{12}^{(k)} = 0$ if k is even). The method used in the previous question shows that there is a unique equilibrium distribution,

$$\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/8 & 3/8 \end{pmatrix}.$$

c) The chain is not irreducible as there is no way of going from state 2 to state 1, for example. You could have said that the absorbing states form single element equivalence classes. Solving the equation

$$\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix},$$

gives that for any $1 \leq \alpha \leq 1$, the vector

$$\begin{pmatrix} 0 & \alpha & 1-\alpha \end{pmatrix}$$

is an equilibrium distribution. So an equilibrium distribution does exist but it is not unique.

4. We know that the graph is finite so let $S = \{1, 2, \ldots, n\}$ be the vertex set of the graph (and hence also the state space of the chain). We will write $\deg(i)$ for the degree of vertex i. The chain has transition probabilities

$$p_{ij} = \begin{cases} \frac{1}{\deg(i)} & \text{if } (ij) \text{ is an edge of the graph} \\ 0 & \text{if } (ij) \text{ is not an edge of the graph} \end{cases}$$

I claim that setting $w_i = \deg(i)$ solves the equation

$$\mathbf{w}P = \mathbf{w}.$$

To see this note that the kth component of the left hand side is

$$\sum_{i=1}^{n} \deg(i) p_{ik}.$$

The summand is 0 if (ik) is not an edge and 1 if ik is an edge (since in that case $p_{ik} = \frac{1}{\deg(i)}$). Since there are $\deg(k)$ values of i for which (ik) is an edge, we deduce that this sum is $\deg(k)$. That is

$$\sum_{i=1}^{n} \deg(i) p_{ik} = \deg(k),$$

and so this $\mathbf{w}$ does indeed solve the equation. It remains to normalise $\mathbf{w}$ so that we have a probability vector. We can do this be dividing every component by $D = \sum_{i=1}^{n} \deg(i)$ (you may have noticed that D is twice the number of edges of the graph). The equilibrium distribution is therefore

$$\mathbf{w} = \begin{pmatrix} \frac{\deg(1)}{D} & \frac{\deg(2)}{D} & \cdots & \frac{\deg(n)}{D} \end{pmatrix}.$$

Please let me know if you have any comments or corrections