Probability III – 2007/08

Solutions to Exercise Sheet 2

1. The transition graph is drawn below:

i,ii) From the transition graph, we can read off the probabilities:

$$\mathbb{P}(X_1 = 2|X_0 = 1) = \frac{1}{2},$$

$$\mathbb{P}(X_2 = 3|X_1 = 2) = \frac{1}{3}.$$

iii) By the Markov property,

$$\mathbb{P}(X_2 = 3|X_1 = 2, X_0 = 1) = \mathbb{P}(X_2 = 3|X_1 = 2) = \frac{1}{3}.$$

iv)

$$\begin{aligned}\mathbb{P}(X_2 = 3, X_1 = 2|X_0 = 1) &= \mathbb{P}(X_2 = 3|X_1 = 2, X_0 = 1) \times \mathbb{P}(X_1 = 2|X_0 = 1) \\ &= \frac{1}{3} \times \frac{1}{2} \\ &= \frac{1}{6}\end{aligned}$$

2. Let state 1 denote a wet day and state 2 denote a dry day. The transition matrix for the chain is then given by

$$P = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}.$$

The 3-step transition matrix is

$$P^3 = \begin{pmatrix} 43/64 & 21/64 \\ 21/32 & 11/32 \end{pmatrix}.$$

i) From the matrix we see that

$$\mathbb{P}(17 \text{ October is dry} | 14 \text{ October is dry}) = \frac{11}{32}.$$

ii) Again, from the matrix,

$$\mathbb{P}(17 \text{ October is wet} | 14 \text{ October is wet}) = \frac{43}{64}.$$

iii)

$$\begin{aligned}\mathbb{P}(17 \text{ and } 18 \text{ October are both wet} | 14 \text{ October is wet}) &= \mathbb{P}(17 \text{ October is wet} | 14 \text{ October is wet}) \\ &\quad \times \mathbb{P}(18 \text{ October is wet} | 17 \text{ October is wet}) \\ &= \frac{43}{64} \times \frac{3}{4} \\ &= \frac{129}{256}\end{aligned}$$

(compare part iv) of question 1.)

3. This question was possibly rather confusingly phrased. Perhaps it would have been clearer to say "draw the transition matrix for a Markov chain which models this process", as the process as described is strictly speaking not defined for all $t \in \{0, 1, 2, \dots\}$.

We must somehow model the fact that the process can end and the most obvious way to do this is to introduce a 6th state called "End" which is absorbing. Doing this we obtain transition graph

and transition matrix

$$\begin{pmatrix} 0 & 1/2 & 0 & 0 & 1/3 & 1/6 \\ 1/3 & 0 & 1/2 & 0 & 0 & 1/6 \\ 0 & 1/3 & 0 & 1/2 & 0 & 1/6 \\ 0 & 0 & 1/3 & 0 & 1/2 & 1/6 \\ 1/2 & 0 & 0 & 1/3 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly, to return to the start after two throws the first two steps must be "clockwise" then "anti-clockwise" or "anti-clockwise" then "clockwise". Thus,

$$\mathbb{P}(\text{return to start after 2 throws}) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{3}.$$

The process lasts for exactly seven throws means that the first six throws are all not 6 and the seventh is a 6. So,

$$\mathbb{P}(\text{lasts 7 throws}) = \frac{5^6}{6^7} \approx 0.0558.$$

4. The absorbing states are 0 and 3 (because $p_{00} = p_{33} = 1$).
Let u_i be the probability that the process is absorbed at state 0 given that it starts at state i. Writing A for the time of absorbtion this means

$$u_i = \mathbb{P}(X_A = 0 | X_0 = i).$$

Using first step analysis (that is to say conditioning on X_1) we obtain the equations

$$u_1 = \frac{1}{4} + \frac{1}{4}u_1 + \frac{1}{4}u_2$$
$$u_2 = \frac{1}{6} + \frac{1}{6}u_1 + \frac{1}{6}u_2.$$

These are easily solved to give $u_1 = \frac{3}{7}$. I didn't ask for u_2 but you could also have found that $u_2 = \frac{2}{7}$.

5. This question was also best tackled using first step analysis.
i) Let w_i be the expected time to absorbtion starting from state i. Writing A for the absorbtion time,

$$w_i = \mathbb{E}(A | X_0 = i).$$

Using first step analysis (see lecture notes) we obtain the equations

$$w_0 = 1 + \frac{1}{7}w_1$$
$$w_1 = 1 + \frac{1}{4}w_0 + \frac{1}{2}w_1.$$

These are easily solved to give $w_1 = \frac{35}{13}$.
ii) Let v_i be the expected number of visits to state 0 before absorbtion starting from state i. Writing A for the absorbtion time. We now use the general method described in lectures taking state 0 to have weight 1 and all others to have weight 0. We obtain the equations

$$v_0 = 1 + \frac{1}{7}v_1$$
$$v_1 = \frac{1}{4}v_0 + \frac{1}{2}v_1.$$

These are easily solved to give $v_1 = \frac{7}{13}$.
iii) Let g_i be your expected gain up to absorbtion starting from state i. Using the same method but this time with weight -5 for state 0 and weight 10 for state 1 we obtain

$$g_0 = -5 + \frac{1}{7} g_1$$
$$g_1 = 10 + \frac{1}{4} g_0 + \frac{1}{2} g_1.$$

These are easily solved to give $g_1 = \frac{245}{13}$.
You could also have done this by noting that

$$\text{Gain} = -5 \times (\text{number of visits to } 0) + 10 \times (\text{numberof visits to } 1).$$

Now using liearity of expectation,

$$g_i = -5 v_i + 10 \mathbb{E}(\text{number of visits to } 1).$$

Which can be found using part ii) and a similar calculation for state 1.

Please let me know if you have any comments or corrections