Probability III – 2007/08

Solutions to Exercise Sheet 1

1. a) We have that

$$\begin{aligned}
\mathbb{P}(X+Y=k) &= \sum_{i=0}^{k} \mathbb{P}(X=i, Y=k-i) \\
&= \sum_{i=1}^{k} \mathbb{P}(X=i)\mathbb{P}(Y=k-i) \qquad \text{(by independence)} \\
&= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \binom{n}{k-i} p^{k-i} (1-p)^{n-k+i} \\
&= p^k (1-p)^{2n-k} \sum_{i=0}^{k} \binom{n}{i}\binom{n}{k-i} \\
&= p^k (1-p)^{2n-k} \binom{2n}{k} \qquad \text{(by standard properties of binomial numbers).}
\end{aligned}$$

This is just the probability mass function of a $\mathrm{Bin}(2n, p)$ random variable.

b) As for part a) we have that

$$\begin{aligned}
\mathbb{P}(X+Y=k) &= \sum_{i=0}^{k} \mathbb{P}(X=i)\mathbb{P}(Y=k-i) \\
&= \sum_{i=0}^{k} e^{\lambda} \frac{\lambda^i}{i!} e^{\lambda} \frac{\lambda^{k-i}}{k-i!} \\
&= \frac{e^{-2\lambda}\lambda^k}{k!} \sum_{i=0}^{k} \frac{k!}{(k-i)!i!} \\
&= \frac{e^{-2\lambda}\lambda^k}{k!} \sum_{i=0}^{k} \binom{k}{i} \\
&= \frac{e^{-2\lambda}\lambda^k}{k!} 2^k \qquad \text{(since the sum is the number of subsets of a } k\text{-set)} \\
&= \frac{e^{-2\lambda}(2\lambda)^k}{k!}
\end{aligned}$$

This is just the probability mass function of a $\mathrm{Po}(2\lambda)$ random variable.

c) We have that

$$\begin{aligned}
\mathbb{P}(X=k|X+Y=n) &= \frac{\mathbb{P}(X=k, X+Y=n)}{\mathbb{P}(X+Y=n)} \qquad \text{(definition of conditional probability)} \\
&= \frac{\mathbb{P}(X=k, Y=n-k)}{\mathbb{P}(X+Y=n)} \\
&= \frac{\mathbb{P}(X=k)\mathbb{P}(Y=n-k)}{\mathbb{P}(X+Y=n)} \qquad \text{(independence)} \\
&= \frac{e^{-\lambda}\frac{\lambda^k}{k!}e^{-\lambda}\frac{\lambda^{n-k}}{(n-k)!}}{e^{-2\lambda}\frac{(2\lambda)^n}{n!}} \qquad \text{(by part b) above)} \\
&= \binom{n}{k}\frac{1}{2^n}
\end{aligned}$$

This is just the probability mass function of a $\text{Bin}(n, \frac{1}{2})$ random variable.

2. a) Since X takes positive integer values,

$$\mathbb{P}(X \geq k) = \mathbb{P}(X=k) + \mathbb{P}(X=k+1) + \mathbb{P}(X=k+2) + \dots$$

Thus,

$$\begin{aligned}
\sum_{k\geq 1}\mathbb{P}(X \geq k) &= (\mathbb{P}(X=1) + \mathbb{P}(X=2) + \mathbb{P}(X=3) + \dots) \\
&\quad + (\mathbb{P}(X=2) + \mathbb{P}(X=3) + \mathbb{P}(X=4) + \dots) \\
&\quad + (\mathbb{P}(X=3) + \mathbb{P}(X=4) + \mathbb{P}(X=5) + \dots) \\
&\quad + \dots \\
&= \sum_{k\geq 1} k\mathbb{P}(X=k) \\
&= \mathbb{E}(X).
\end{aligned}$$

b) The event $(N=n)$ happens precisely when the first $n-1$ tosses are all tails and the next one is a head. Thus,

$$\mathbb{P}(N=n) = \left(\frac{1}{2}\right)^{n-1}\frac{1}{2} = \left(\frac{1}{2}\right)^n.$$

This is the geometric distribution with parameter $\frac{1}{2}$.
To find the expectation observe that the event $(N \geq k)$ happens precisely when the first $k-1$ tosses are all tails. Thus,

$$\mathbb{P}(N \geq k) = \left(\frac{1}{2}\right)^{k-1}.$$

Using part a) we obtain that

$$\begin{aligned}
\mathbb{E}(N) &= \sum_{k\geq 1}\left(\frac{1}{2}\right)^{k-1} \\
&= 2 \qquad \text{(sum of a geometric progression).}
\end{aligned}$$

3. Let X be the number of heads observed. Conditioning on N we obtain

$$\begin{aligned}\mathbb{P}(X=k) &= \sum_{n\geq k}\mathbb{P}(X=k|N=n)\mathbb{P}(N=n)\\ &= \sum_{n\geq k}\binom{n}{k}p^k(1-p)^{n-k}e^{-\lambda}\frac{\lambda^n}{n!} = \frac{e^{-\lambda}p^k\lambda^k}{k!}\sum_{n\geq k}\frac{(1-p)^{n-k}\lambda^{n-k}}{(n-k)!}\\ &= \frac{e^{-\lambda}(p\lambda)^k}{k!}e^{(1-p)\lambda} = \frac{e^{-p\lambda}(p\lambda)^k}{k!}.\end{aligned}$$

Hence X is distributed $\mathrm{Po}(p\lambda)$.

4. Solution 1. If ξ_j is the random number chosen at j-th selection, then $X=\xi_1+\cdots+\xi_k$. But then

$$\begin{aligned}\mathbb{P}(\xi_j=i) &= \mathbb{P}(\xi_1\neq i,\ldots,\xi_{j-1}\neq i,\xi_j=i) \quad (\text{since } \mathbb{P}(AB)=\mathbb{P}(A)\mathbb{P}(B|A))\\ &= \mathbb{P}(\xi_1\neq i,\ldots,\xi_{j-1}\neq i)\mathbb{P}(\xi_j=i\,|\,\xi_1\neq i,\ldots,\xi_{j-1}\neq i) = \frac{\binom{n-1}{j-1}}{\binom{n}{j-1}}\frac{1}{n-j+1} = \frac{1}{n}.\end{aligned}$$

Hence,

$$\mathbb{E}(\xi_j) = \frac{1}{n}\Big(1+2+3+\cdots+n\Big) = \frac{n+1}{2} \quad (\text{sum of an arithmetic progression})$$

and

$$\mathbb{E}(X) = \mathbb{E}(\xi_1)+\cdots+\mathbb{E}(\xi_k) = \frac{k(n+1)}{2}.$$

Solution 2. Let A_i be the event "the ith ball is chosen". Using indicator functions we see that

$$X = \mathbb{1}(A_1)+2\mathbb{1}(A_2)+3\mathbb{1}(A_3)+\cdots+n\mathbb{1}(A_n)$$

Now by the linearity of expectation

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{1}(A_1))+2\mathbb{E}(\mathbb{1}(A_2))+3\mathbb{E}(\mathbb{1}(A_3))+\cdots+n\mathbb{E}(\mathbb{1}(A_n)).$$

It is clear that for any event S, $\mathbb{E}(\mathbb{1}(S))=\mathbb{P}(S)$. Also, for any i,

$$\mathbb{P}(A_i) = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}.$$

Thus,

$$\mathbb{E}(X) = \frac{k}{n}\Big(1+2+3+\cdots+n\Big) = \frac{k(n+1)}{2} \quad (\text{sum of an arithmetic progression}).$$

5. Let's work out the transition probabilities for all pairs of states. It is clear that

$$\mathbb{P}(X_{n+1} = j | X_n = i) = 0 \qquad \text{if } i > j.$$

If $X_n = i$ then $X_{n+1} = i$ if and only if throw number $(n+1)$ is no more than i. Thus,

$$\mathbb{P}(X_{n+1} = i | X_n = i) = \frac{i}{6}.$$

If $i < j$, and $X_n = i$ then $X_{n+1} = j$ if and only if throw number $(n+1)$ is a j. Thus,

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \frac{1}{6} \qquad \text{if } i < j.$$

Since these transition probabilities do not depend on $X_{n-1}, X_{n-2}, \ldots$ we deduce that X_n is a Markov chain. The transition graph is drawn below

If throw number $(n+1)$ is not a 6 then he number of 6s seen on the first $(n+1)$ rolls is the number of sixes seen in the first n rolls. If throw number $(n+1)$ is a 6 then it increases by 1. Thus

$$\mathbb{P}(S_{n+1} = j | S_n = i) = \begin{cases} \frac{5}{6} & \text{if } i = j \\ \frac{1}{6} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Again, these transition probabilities do not depend on $S_{n-1}, S_{n-2}, \ldots$. We deduce that X_n is a Markov chain. The transition graph is drawn below

Finally, T_n is not a Markov chain. To show this we must show that the Markov condition is violated. Suppose that $T_1 = 1, T_2 = 1$; in this case the first throw must have been a 6 and the second throw must have been something else. Thus T_3 is at most 1. It follows that

$$\mathbb{P}(T_3 = 2 | T_2 = 1, T_1 = 1) = 0.$$

Suppose that $T_1 = 0, T_2 = 1$; in this case the first throw must have been something other than a 6 and the second throw must have been a 6. Thus, if the third throw is a 6 then $T_3 = 2$. It follows that,

$$\mathbb{P}(T_3 = 2 | T_2 = 1, T_1 = 0) = \frac{1}{6}.$$

We deduce that $\mathbb{P}(T_3 = 2 | T_2 = 1)$ does depend on T_1 and hence the Markov property does not hold.

Please let me know if you have any comments or corrections