MAS338 Probability III

The purpose of this list of questions is to help you to prepare for the exam by revising the material of the course in a systematic way. Apart of the ability to provide answers to these questions you are supposed to be able to solve problems similar to those considered in courseworks.

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1. The notion of a Markov Chain (MC).
   (a) What is the definition of a Markov Chain?
   (b) What is a transition matrix of a Markov Chain?
   (c) Let X_n be a Markov Chain with a state space $S = \{1, \dots, m\}$ and a transition matrix $\mathbb{P} = (p_{ij})$. Prove that
   $$P\{X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_1 = i_1 \,|\, X_0 = i_0\} = p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{n-1} i_n}.$$
   (d) Suppose that the initial distribution of a Markov Chain with a state space S is given: $P\{X_0 = i\} = \pi_i$, where $i \in S$. Express the probability $P\{X_2 = k\}$ in terms of $p_{ij},\ i, j \in S$, and π_i.
   (e) Prove the following theorem:
   Theorem. *If X_n is a Markov Chain with a state space $S = \{1, \dots, m\}$ and a transition matrix $\mathbb{P} = (p_{ij})$, then $P\{X_n = j \,|\, X_0 = i\} = p_{ij}^{(n)}$, where $p_{ij}^{(n)}$ is the (i, j)-element of the matrix $\mathbb{P}^n$.*

2. First step analysis.
   (a) Define what is an absorbing state of a Markov Chain.
   (b) Consider a Markov Chain with a state space $S = \{0, 1, 2\}$ and a transition matrix
   $$P = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{pmatrix}.$$
   (a) Explain why 0 and 2 are absorbing states of this chain.
   (b) Let T be the time at which the chain is absorbed: $T = \min\{\, n : X_n = 0 \text{ or } X_n = 2 \,|\, X_0 = 1\}$. Put $u = P\{X_T = 0 \,|\, X_0 = 1\}$ – the probability that the chain is absorbed by the state 0. Using the first step analysis derive the equation satisfied by u and hence find u.
   How would the answer change if you are asked to find $\tilde{u} = P\{X_T = 2 \,|\, X_0 = 1\}$?
   (c) Let T be as above. Put $v = E\{T \,|\, X_0 = 1\}$ – the (conditional) expectation of the absorbtion time. Using the first step analysis derive the equation satisfied by v and hence find v.

3. Consider a Markov Chain with a state space $S = \{1, 2, 3, 4\}$ and transition matrix

$$P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

State questions analogous to the ones asked in the previous problem and answer them.

4. Consider a Markov Chain with a state space $S = \{1, 2, 3, 4\}$ and transition matrix given above. Denote by a_i, $i = 2,\ 3$, the probability of the event that a MC starting from i will eventually be absorbed by one of the absorbing states. Use the first step analysis to derive equations for these probabilities and prove that $a_i = 1$.

5. Consider a Markov Chain with a state space $S = \{1, 2, 3, 4\}$ and transition matrix given above. Let u_i be the probability that a MC starting from i is absorbed by the state 4. Find these probabilities.

6. Consider a Markov Chain with a state space $S = \{1, 2, 3, 4\}$ and transition matrix given above. Let $v_i = E\{T \mid X_0 = i\}$ be the mean time of absorbtion given that the MC starts from i. Find these expectations.

7. Consider a Markov Chain with a state space $S = \{1, ..., m\}$ and transition matrix $\mathbb{P} = (p_{ij})$. Suppose that this chain has absorbing states and let T be the time at which the chain is absorbed by one of these states. Next, let $f(i)$ be a function defined on S. Put

$$w_i = E(\sum_{j=0}^{T} f(X_j) \mid X_0 = i).$$

Prove that

$$\begin{aligned} w_i =& f(i) \ \text{ if } i \text{ is absorbing} \\ w_i =& f(i) + \sum_{j=1}^{m} p_{ij} w_j \ \text{ if } i \text{ is not absorbing.} \end{aligned}$$

How should one choose the function f in order to obtain equations for u_i? For v_i?

8. A gambler is playing the following game. A fair 6-sided die is rolled repeatedly until the sum of two consecutive throws is 3 for the first time. If 6 is rolled then the gambler is paid £1; if 2 is rolled then the gambler pays £1. The gambler neither receives nor pays any money if anything else is rolled. However, he does not pay anything if 2 is rolled at the very end of the game.

Consider a Markov chain X_n with the state space $S = \{0, 6, 1, 2, 12, 21\}$, where 0 denotes the beginning of the game (that is $X_0 = 0$) and it reappears every time when none of the numbers 1, 2, 6 shows up. The other states have a natural meaning: 6, 1, and 2 are the values of X_n whenever they appear strictly before the end of the game, whereas 12 and 21 denote the end of the game.

(a) Write down the transition matrix for this Markov chain.

(b) What is the expected number of rolls in this game?

(c) Obtain the mean value of the gain in this game and decide whether this game is fair.

(d) If it turns out that this game is not fair, what should be the amount £z paid for rolling a 6 to ensure that the modified game is fair. (No other payment is changed.)

9. Define what is an equilibrium distribution of a Markov Chain X_t with a transition matrix $\mathbb{P} = (p_{ij})$, $1 \le i,\ j \le m$.

   Suppose that the initial distribution of the MC X_t is π, where $\pi = (\pi_1, ..., \pi_m)$ is an equilibrium distribution of this MC. What is then the probability $P\{X_5 = i\}$? What is the probability $P\{X_5 = i,\ X_6 = j\}$?

10. Give the definition of an irreducible MC. What is a regular MC? Is a regular chain irreducible? Is an irreducible chain regular? (Wherever the answer is NO, give an example illustrating this answer.)

    Suppose that the state space of a MC is finite. State a theorem providing a sufficient condition for existence of a unique equilibrium distribution of such a chain.

11. Consider a MC with an infinite state space $S = \{0, 1, 2, ...\}$. Suppose that the transition probabilities of this MC are given by

$$p_{i,0} = q, \quad p_{i,i+1} = p, \quad \text{where } p > 0,\ q > 0,\ p + q = 1$$

    (obviously, $p_{ij} = 0$ if $j \neq i + 1$ or 0). Find the equilibrium distribution for this MC.

12. If a finite MC is irreducible and $p_{ii} > 0$ for some i, then it is regular.

    Every finite regular MC has a unique equilibrium distribution.

    Use these two facts to prove the following

    Theorem. *An irreducible MC with a finite state space has a unique equilibrium distribution.*

13. The Law of Large Numbers for a finite MC.

    (a) Let $f : S \mapsto \mathbb{R}$ be a function on a the state space of our MC. Suppose that the initial distribution of the MC X_n is $\underline{\mu} = (\mu_1, ..., \mu_m)$, that is $P\{X_0 = i\} = \mu_i$. Show that then $Ef(X_n) = \sum_{j=1}^{m} \mu_j^{(n)} f(j)$, where $\mu_j^{(n)} = \sum_{j=1}^{m} \mu_i p_{ij}^{(n)}$.

    (b) State the Law of Large Numbers for a finite MC.

    (c) Use the LLN to explain the connection between the number of visits to a state i and π_i, where $\underline{\pi} = (\pi_1, ..., \pi_m)$ is the equilibrium distribution of an irreducible MC.

14. Recurrence.

(a) Give the definition of a recurrent state of a Markov chain.

(b) Let X_n be a Markov chain with a state space S and transition probabilities $\mathbb{P} = (p_{ij})_{i,j\in S}$. State the theorem which provides the necessary and sufficient condition for a state $i \in S$ to be recurrent in terms of the probabilities $p_{ii}^{(n)}$.

c) We say that a MC is recurrent if all states of this chain are recurrent. In view of this definition:

Is a finite regular MC recurrent? Explain your answer.

Is an irreducible finite MC recurrent? Explain your answer.

(d) Put $f_i^{(n)} = P\{X_n = i, X_{n-1} \neq i, ..., X_1 \neq i \,|\, X_0 = i\}$ – the probability that the first return to i of a MC starting from i happens at time n.

Prove that

$$p_{ii}^{(n)} = \sum_{k=1}^{n-1} f_i^{(k)} p_{ii}^{(n-k)} + f_i^{(n)}.$$

Derive from here recursive formulae for $f_i^{(n)}$, $n = 1, 2, ...$

What is the property of the sequence $f_i^{(n)}$ implying recurrence of i?

(e) Put $\beta_i = \sum_{k=1}^{\infty} f_i^{(k)}$. What is the probabilistic meaning of β_i?

Let M be the number of returns to i. Prove the following statements

Lemma 1. *Suppose that $\beta_i < 1$. Then $P\{M \geq k \,|\, X_0 = i\} = \beta_i^k$, where $k = 1, 2....$*

Lemma 2. *Suppose that $\beta_i < 1$. Then $P\{M = k \,|\, X_0 = i\} = \beta_i^k - \beta_i^{k+1}$, where $k = 0, 1, 2....$*

Remark. Lemmas 1, 2 are correct also when $\beta_i = 1$; in this case they are very simple.

Lemma 3. *If i is non-recurrent then $E(M) = \frac{\beta_i}{1-\beta_i}$.*

Theorem. *$\beta_i = 1$ if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$. Equivalently, $\beta_i < 1$ if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$.*

(In other words, i is recurrent if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$.)

15. Prove that if i and j intercommunicate, then i is recurrent if and only if j is recurrent.

16. State the Theorem (the so called basic limit theorem of the theory of Markov chains) which establishes the relation between the $\lim_{n\to\infty} p_{ii}^{(n)}$ and the expectation of the return time R_i of the MC, starting from state i, to i: $E(R_i \,|\, X_0 = i) \equiv \sum_{k=1}^{\infty} k f_i^{(k)}$.

17. Poisson Processes.

   (a) What is a Poisson random variable ξ with parameter μ? Prove that $E\xi = \mu$, $\mathrm{Var}\xi = \mu$. Prove that a sum of two independent r. v.'s having Poisson distribution is a Poisson r. v.

(b) Give the axiomatic definition of a Poisson process.

(c) Give the infinitesimal definition of a Poisson process.

(d) Prove the following

Theorem. *Suppose that $X(t)$ is random process satisfying the conditions of the infinitesimal definition of a Poisson process. Then $P\{X(t) = k\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$, where $k = 0, 1, 2....$*

Remark. You are supposed to prove that $p'_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t)$, where $p_k(t) \overset{\text{def}}{=} P\{X(t) = k\}$ and to check that $p_k(t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$ satisfies these equations as well as the relevant initial conditions.

18. Distributions associated with a Poisson process.

(a) Define occurrence times W_j and sojourn times S_j for a Poisson process $X(t)$.

(b) Prove the following theorems.

Theorem 1. *The sojourn times S_j, $j = 1, 2, ...$, form a sequence of independent identically distributed random variables whose density function is given by $f_{S_j}(t) = \lambda e^{-\lambda t}$, where $t \geq 0$.*

Theorem 2. *The occurrence times W_n, $n = 1, 2, ...$ are random variables whose probability density function is given by $f_{W_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$, where $t \geq 0$.*

Theorem 3. *Suppose that $X(t)$ is a Poisson process and $0 < u < t$. Then $P\{X(u) = k \,|\, X(t) = n\} = \binom{n}{k}(\frac{u}{t})^k(1 - \frac{u}{t})^{n-k}$, where $0 \leq k \leq n$.*

(c) Let $W_1, ..., W_n$ be the event times of a Poisson process. Given that $X(t) = n$, what is the joint probability density function of random variables $W_1, ..., W_n$? State the corresponding theorem and write down the formula for

$f_{\{W_1,...,W_n|X(t)=n\}}(x_1, ...x_n)$.

Exercise. Prove this formula for the cases $n = 1$ and $n = 2$.

(d) Suppose that $R(W_1, ..., W_n)$ is a symmetric function of W_i. State the theorem allowing one to evaluate $E(R(W_1, ..., W_n) \,|\, X(t) = n)$ in terms of the uniform distribution on $[0, t]$.

19. Birth Process.

(a) Define what is a birth process.

(b) Prove the following

Theorem 1. *Suppose that $X(t)$ is a birth process with $X(0) = 0$. Set $p_n(t) \overset{\text{def}}{=} P\{X(t) = n | X(0) = 0\}$ Then the functions $p_n(t)$ satisfy the following equations:*

$$\begin{cases} p'_n(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t) & n \geq 0 \\ p_0(0) = 1 & \\ p_n(0) = 0 & \text{if } n > 0. \end{cases} \tag{1}$$

(c) Prove the following

Theorem 2. *Equations (1) have a unique solution which can be obtained recursively using the following formulae:*

$$\begin{cases} p_0(t) = e^{-\lambda_0 t} \\ p_n(t) = \lambda_{n-1} e^{-\lambda_n t} \int_0^t e^{\lambda_n s} p_{n-1}(s) ds \quad n > 0 \end{cases} \tag{2}$$

20. Birth and Death Process.

(a) Define what is a Birth and Death process. What is the infinitesimal generator of a Birth and death process?

(b) Let S_i be the time the birth and death process, starting from i, spends in state i. Describe a typical trajectory of a B&D process in terms of random variables S_i and birth and death parameters $\lambda_i,\ \mu_i$ in the case when the birth parameters $\lambda_i > 0$ for all $i \geq 0$ and the death parameters $\mu_i > 0$ for $i > 0$ (as usual, $\mu_0 = 0$).

(c) Prove that S_i is an exponential random variable with parameter $\lambda_i + \mu_i$.

(d) State what are the backward and forward Kolmogorov equations related to a B&D process.

(e) What is the equilibrium distribution of a B&D process. Prove that

$$w_j = \frac{\lambda_0 \lambda_1 \ldots \lambda_{j-1}}{\mu_1 \mu_2 \ldots \mu_j} w_0.$$

(f) Prove that if $\lambda_i = \lambda > 0$ for $i \geq 0$, $\mu_i = \mu > 0$ for $i \geq 1$, and $\lambda < \mu$, then the equilibrium distribution of this B&D process is given by $w_j = (\frac{\lambda}{\mu})^j (1 - \frac{\lambda}{\mu})$. (In other words, the equilibrium distribution is the geometric one.)