BSc examination by course unit

MTH6130 Probability III

Date 2009 Time **Duration 2 hours**

You should attempt ALL questions. Marks awarded are shown next to the question.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

You are not allowed to start reading the question paper until instructed to do so by the invigilator

THIS IS A SAMPLE PAPER

1. Markov Chains

2 (a) What is the definition of a Markov Chain?

10 (b) Prove the following theorem:

Theorem. *If X_n is a Markov Chain with a state space $S = \{1, \dots, m\}$ and a transition matrix $\mathbb{P} = (p_{ij})$, then $P\{X_n = j \mid X_0 = i\} = p_{ij}^{(n)}$, where $p_{ij}^{(n)}$ is the (i, j)-element of the matrix $\mathbb{P}^n$.*

Hint. Denote by $\mathbb{P}^{(n)} = \left(p_{ij}^{(n)}\right)$ and find the relation between $\mathbb{P}^{(n)}$ and $\mathbb{P}^{(n-1)}$

2. First step analysis.

An urn contains 4 red balls and 3 green balls. A ball is chosen at random with each ball in the urn equally likely to be chosen. If the chosen ball is red then it is removed. If the chosen ball is green then it is replaced. The process is repeated until all the red balls have been removed.

Let X_n denote the number of red balls in the urn after n steps.

5 (a) Prove that X_n is a Markov chain with a state space $S = \{0, 1, 2, 3, 4\}$ and write down the corresponding transition matrix.

3 (b) How is an absorbing state of a general Markov Chain defined? Which state of the above Markov chain is absorbing?

10 (c) Find the expected duration of the above process.

Hint. Let $w_i = \mathbb{E}(\text{time to absorbtion}|X_0 = i)$ and use first step analysis to derive the equations for w_i.

3. The equilibrium distribution of a Markov chain.

3 (a) Define what is meant by an equilibrium distribution of a Markov chain X_t with a transition matrix $\mathbb{P} = (p_{ij})$, $1 \leq i,\ j \leq m$.

12 (b) What is a regular Markov chain?

Consider a Markov chain with a state space $S = \{1, 2, 3, 4\}$ and a transition matrix

$$\mathbb{P} = \begin{pmatrix} 1/10 & 1/2 & 0 & 2/5 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Explain why this chain is regular.

Find the limiting distribution $\underline{w} = (w_1, ..., w_4)$ of this chain.

What proportion of time do you expect the chain to spend in each state in the long run?

5 (c) Suppose that the initial distribution of the Markov chain X_t defined in part (b) is the $\underline{w}$ (found in (b)). Find the following probabilities: $P\{X_4 = 2\}$, $P\{X_4 = 2,\ X_5 = 2\}$, $P\{X_4 = 2,\ X_5 = 4\}$. Give a brief explanation to your answers.

4. Poisson processes and associated distributions.

5 (a) Give the axiomatic definition of a Poisson process $X(t)$.

5 (b) Define occurrence times W_j and sojourn times S_j for a Poisson process.

9 (c) Prove the following statements:

Theorem 1. *The occurrence times W_n, $n = 1, 2, \ldots$ are random variables whose probability density function is given by $f_{W_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$, where $t \geq 0$.*

6 (d) **Theorem 2.** *If $X(t)$ is a Poisson process and $0 < u < t$, then $P\{X(u) = k \mid X(t) = n\} = \binom{n}{k}(\frac{u}{t})^k(1 - \frac{u}{t})^{n-k}$, where $0 \leq k \leq n$. In other words, the distribution of $x(u)$ conditioned on $x(t) = n$ is $\mathrm{Bin}(n, \frac{u}{t})$.*

Remark. In (c) and (d) you are supposed to use (without proof) the fact that the probabilities $p_k(t) \stackrel{\text{def}}{=} P\{X(t) = k\}$ are given by $p_k(t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$.

5. Pure death processes.

5 (a) Define what is a birth and death process.

16 (b) Consider a death process $X(t)$ with parameters $\mu_0 = 0$ and $\mu_i > 0$, $i = 1, 2, 3, \ldots$. Suppose that $X(0) = N$ and set $p_n(t) \stackrel{\text{def}}{=} P\{X(t) = n\}$, where $N \geq n \geq 0$. Prove that

$$
\begin{aligned}
p_N'(t) &= -\ \mu_N p_N(t) \\
p_n'(t) &= -\ \mu_n p_n(t) + \mu_{n+1} p_{n+1}(t) \quad \text{if} \quad 0 \leq n \leq N-1
\end{aligned}
$$

4 (c) Using the above equations and the fact that $p_N(0) = 1$, prove that

$$p_N(t) = e^{-\lambda_N t}.$$

End of examination.