The Uniform Distribution and the Poisson Process

1 Definitions and main statements

Let $X(t)$ be a Poisson process of rate λ. Let $W_1, W_2, ..., W_n$ be the event (the occurrence, or the waiting) times.

Question: What is the joint distribution of $W_1, W_2, ..., W_n$ conditioned on the event $X(t) = n$.

It turns out that to **answer** this question it is convenient to introduce a sequence $U_1, U_2, ..., U_n$ of independent random variables which are uniformly distributed on $[0, T]$. Remember that this by definition means that the joint p.d.f of $U_1, U_2, ..., U_n$ is

$$f_{U_1,...,U_n}(x_1, x_2, ..., x_n) = f_{U_1}(x_1)\cdots f_{U_n}(x_n) = \begin{cases} \frac{1}{t^n} & \text{if } 0 \le x_i \le t \text{ for all } i,\ 1 \le i \le n, \\ 0 & \text{otherwise,} \end{cases}$$

where $f_{U_j}(x) = \begin{cases} \frac{1}{t} & \text{if } 0 \le x \le t \\ 0 & \text{otherwise} \end{cases}$. Once a sequence of random variables $U_1, U_2, ..., U_n$ is given, we can re-arrange them into a growing sequence $\tilde{W}_1 < \tilde{W}_2 < ... < \tilde{W}_n$ thus obtaining a new sequence of random variables. The remarkable fact is that

The joint distribution of $W_1, W_2, ..., W_n$ conditioned on the event $X(t) = n$ coincides with the joint distribution of $\tilde{W}_1, \tilde{W}_2, ..., \tilde{W}_n$.

It should be appreciated that this statement allows one to replace the *conditional distribution* of $W_1, W_2, ..., W_n$ by a *distribution* of a relatively simple sequence of random variables $\tilde{W}_1, \tilde{W}_2, ..., \tilde{W}_n$.

The following theorem is in fact a quantitative version of the above statement:

Theorem 1.1
The joint probability density function of $W_1, W_2, ..., W_n$ conditioned on the event $X(t) = n$ is given by

$$f_{\{W_1,...,W_n|X(t)=n\}}(x_1, x_2, ..., x_n) = \begin{cases} \frac{n!}{t^n} & \text{if } 0 \le x_1 \le x_2 \le ... \le x_n \le t, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Remarks and exercises. 1. In the case $n = 1$ Theorem 1.1 states that the event time W_1 is uniformly distributed on $[0, t]$ given that the total number of events observed by time t is 1. *Exercise:* Prove Theorem 1.1 for $n = 1$ (this is very easy to do).

2. If $n = 2$, formula (1) defines a uniform distribution on a triangle in the (x_1, x_2)-plane. *Exercise:* Draw the picture of this triangle. Prove Theorem 1.1 for $n = 2$ (this is slightly more difficult than when $n = 1$).

The random variables U_j are particularly helpful when one wants to find the expectation of $R(W_1, ..., W_n)$, where $R(\cdot)$ is a symmetric function.

Remember that $R(x_1, ..., x_n)$ is said to be a symmetric function if $R(x_1, ..., x_n) = R(x_{i_1}, ..., x_{i_n})$ for any permutation $(i_1, ..., i_n)$ of the sequence $(1, ..., n)$.

Theorem 1.2
Suppose that $W_1, W_2, \ldots W_n$ are occurrence times of a Poisson process of rate $\lambda > 0$. Let $U_1, U_2, \ldots, U_n$ be a sequence of independent random variables which are uniformly distributed on $[0, t]$. Let $R(W_1, \ldots, W_n)$ be a symmetric function of n variables. Then

$$E\left(R(W_1, \ldots, W_n) | X(t) = n\right) = E\left[R(U_1, \ldots, U_n)\right]. \tag{2}$$

2 Extended example

Customers arrive at a facility as a Poisson Process of rate λ. There is a waiting cost of £C per person per unit of time. Customers gather at the facility and are dispatched at time T irrespective of the number of customers. Each dispatch costs £k (irrespective of the number of customers).
Question: How should T be chosen to minimize the expected cost per unit of time?

Solution. First of all

$$\text{dispatching cost per unit of time } = \frac{k}{T}.$$

Next

$$E(\text{waiting cost per unit of time}) =$$
$$\frac{1}{T}\sum_{n=0}^{\infty} E\left[C(T - W_1) + C(T - W_2) + ...C(T - W_n) | X(t) = n\right] P\{X(T) = n\}.$$

By Theorem 1.2,

$$E\left[C(T - W_1) + C(T - W_2) + ...C(T - W_n) | X(t) = n\right] = CE\left[nT - \sum_{i=1}^{n} U_i\right] = Cn\frac{T}{2}.$$

Hence

$$\begin{aligned} E(\text{waiting cost per unit of time}) =& \frac{1}{T}\sum_{n=0}^{\infty} Cn\frac{T}{2} \times P\{X(T) = n\} \\ =& \frac{C}{2}\sum_{n=0}^{\infty} ne^{-\lambda T}\frac{(\lambda T)^n}{n!} = \frac{C\lambda T}{2}. \end{aligned}$$

Thus

$$E(\text{total cost per unit of time}) = \frac{C\lambda T}{2} + \frac{k}{T} = f(T).$$

But then $f'(T) = \frac{C\lambda}{2} - \frac{k}{T^2} = 0$ implies $T = \sqrt{\frac{2k}{C\lambda}}$.
Example 2. Study very carefully Problem 5 from CW6.