THE EIGENVALUE PROBLEM FOR A CLASS OF LINEAR INTEGRAL OPERATORS. MAS214: LODE

Let $L = \{f(x),\ a \le x \le b\}$ be the space of continuous functions on $[a, b]$. A linear integral operator A with kernel $K(x, y)$ is defined by

$$(Af)(x) = \int_a^b K(x, y)f(y)dy,$$

where $K(x, y)$ is a continuous function of $(x, y),\ a \le x, y \le b$.
Definition 1. f is an eigenfunction of A if $Af = \lambda f$, and λ is the eigenvalue corresponding to the eigenfunction f.
In our case the eigenvalue-eigenfunction equation reads

$$(Af)(x) = \int_a^b K(x, y)f(y)dy = \lambda f(x),$$

where $f(x)$ and λ are the unknowns. However, the task of solving this equation with arbitrary $K(x, y)$ is far too difficult if not impossible. In these notes we consider a very particular but useful case when $K(x, y)$ is a finite sum of products of continuous functions $\phi_j(x)g_j(y)$:

$$K(x, y) = \sum_{j=1}^{n} \phi_j(x)g_j(y). \quad (1)$$

We shall make use of the following notations:

$$a_{jk} = \int_a^b g_j(y)\phi_k(y)dy$$

Theorem 1. *Suppose that functions $\phi_j(x),\ 1 \le j \le n$ are linearly independent. Then*
(i) $f(x)$ is the eigenfunction with zero eigenvalue if and only if

$$\int_a^b g_j(y)f(y)dy = 0 \text{ for all } j,\ 1 \le j \le n. \quad (2)$$

(ii) $\lambda \ne 0$ is a non-zero eigenvalue of A if and only if it satisfies the following equation:

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12}\dots & a_{1n} \\ a_{21} & a_{22} - \lambda\dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{nn} - \lambda \end{pmatrix} = 0 \quad (3)$$

and the relevant eigenfunction is of the form

$$f(x) = \sum_{j=1}^{n} c_j\phi_j(x),$$

where $c_1, ..., c_n$ is a (non-trivial) solution of the following system of linear equations:

$$\begin{cases} (a_{11}-\lambda)c_1 + a_{12}c_2 + ... + a_{1n}c_n = 0 \\ a_{21}c_1 + (a_{22}-\lambda)c_2 + ... + a_{2n}c_n = 0 \\ \qquad\qquad ... \\ a_{n1}c_1 + a_{n2}c_2 + ... + (a_{nn}-\lambda)c_n = 0 \end{cases} \tag{4}$$

Remarks. 1. The easy consequence of this theorem is that zero is always among the finite number of the eigenvalues of our operator. Indeed, relations (2) tells us that any function $f(x)$ which is orthogonal to all functions $\overline{g_j(x)}$ is an eigenfunction of this operator with eigenvalue 0. The existence of (infinitely many) such functions follows for instance from the orthogonalization procedure.

2. Since (3) is a polynomial equation and the polynomial in the lhs of (3) is of degree n, the operator A has no more than n non-zero eigenvalues in total.

3. If the only solution to (3) is zero, then 0 is the only eigenvalue of A.

Proof of Theorem 1.

(i) If $f(x)$ is an eigenfunction with zero eigenvalue then by definition

$$(Af)(x) = \int_a^b K(x,y)f(y)dy = 0$$

or

$$\int_a^b K(x,y)f(y)dy = \sum_{j=1}^n \phi_j(x) \int_a^b g_j(y)f(y)dy) = 0 \text{ for all(!) } x.$$

Since functions $\phi_j(x)$ are linearly independent the last equation holds if and only if all relations (2) are satisfied. QED

(ii) If $\lambda \neq 0$ is an eigenvalue of A, then

$$(Af)(x) = \int_a^b K(x,y)f(y)dy = \lambda f(x)$$

or

$$\int_a^b K(x,y)f(y)dy = \sum_{j=1}^n \phi_j(x) \int_a^b g_j(y)f(y)dy) = \lambda f(x). \tag{5}$$

Hence

$$f(x) = \lambda^{-1} \sum_{j=1}^n \phi_j(x) \int_a^b g_j(y)f(y)dy) = \sum_{j=1}^n c_j \phi_j(x), \tag{6}$$

where $c_j = \lambda^{-1} \int_a^b g_j(y)f(y)dy$ which can also be re-written as

$$\int_a^b g_j(y)f(y)dy = \lambda c_j \tag{7}$$

Note that at this stage we don't know whether the c_j indeed exist; but we do already know that, if there is an eigenfunction with a non-zero eigenvalue, then it has the above form. Substituting now (6) into (7), we obtain:

$$\int_a^b g_j(y) \sum_{k=1}^n c_k \phi_k(y) dy) = \lambda c_j \text{ for all } j, \ 1 \leq j \leq n \tag{8}$$

or

$$\sum_{k=1}^n c_k \int_a^b g_j(y) \phi_k(y) dy) = \lambda c_j \text{ for all } j, \ 1 \leq j \leq n \tag{9}$$

Finally, taking into account that we denoted $a_{jk} = \int_a^b g_j(y)\phi_k(y)dy$, we can rewrite (9) as

$$\begin{cases} (a_{11} - \lambda)c_1 + a_{12}c_2 + ... + a_{1n}c_n = 0 \\ a_{21}c_1 + (a_{22} - \lambda)c_2 + ... + a_{2n}c_n = 0 \\ \qquad \qquad \qquad ... \\ a_{n1}c_1 + a_{n2}c_2 + ... + (a_{nn} - \lambda)c_n = 0 \end{cases}$$

which finishes the proof of the 'if' direction of statement (ii). The 'only if' direction is now almost obvious: one simply has to note that in fact the implications at each step can be reversed. QED

Example. A linear integral operator A with kernel $K(x, y) = x^2 + y^2$ acts on the space of continuous functions $L = \{f(x), \ 0 \leq x \leq 1\}$ in the usual way:

$$(Af)(x) = \int_0^1 (x^2 + y^2) f(y) dy.$$

Find all eigenvalues of this operator. Also find all eigenfunctions corresponding to non-zero eigenvalues.

Solution. 1. In our case

$$K(x, y) = \phi_1(x)g_1(y) + \phi_2(x)g_2(y)$$

where

$$\phi_1(x) = x^2, \ g_1(y) = 1, \ \phi_2(x) = 1, \ g_2(y) = y^2.$$

2. Hence we can find the coefficients $a_{jk} == \int_0^1 g_j(y)\phi_k(y)dy$:

$$a_{11} = \int_0^1 g_1(y)\phi_1(y)dy = \int_0^1 y^2 dy = \frac{1}{3}, \quad a_{12} = \int_0^1 g_1(y)\phi_2(y)dy = \int_0^1 dy = 1$$

$$a_{21} = \int_0^1 g_2(y)\phi_1(y)dy = \int_0^1 y^4 dy = \frac{1}{5}, \quad a_{22} = \int_0^1 g_2(y)\phi_2(y)dy = \int_0^1 y^2 dy = \frac{1}{3}$$

3. To find the non-zero eigenvalues we have to solve the equation

$$\det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} = \det\begin{pmatrix} \frac{1}{3}-\lambda & 1 \\ \frac{1}{5} & \frac{1}{3}-\lambda \end{pmatrix} = 0 \tag{10}$$

or equivalently

$$(\frac{1}{3}-\lambda)^2 - \frac{1}{5} = 0.$$

The solutions to this equation are

$$\lambda_1 = \frac{1}{3} + \frac{1}{\sqrt{5}}, \quad \lambda_2 = \frac{1}{3} - \frac{1}{\sqrt{5}}$$

and we thus found the no-zero eigenvalues.

4. We know that the eigenfunctions related to the non-zero eigenvalues are given by

$$f(x) = c_1\phi_1(x) + c_2\phi_2(x)$$

where $(c_1,\ c_2)$ is any(!) non-trivial (that is non-zero) solution to

$$(a_{11}-\lambda)c_1 + a_{12}c_2 = 0.$$

In our case with $\lambda = \lambda_1$ we have:

$$-\frac{1}{\sqrt{5}}c_1 + c_2 = 0, \quad \text{or} \quad c_1 = c_2\sqrt{5},$$

where c_2 is any non-zero number. The corresponding eigenfunction is

$$f_1(x) = c_2\sqrt{5}x^2 + c_2 = c_2(\sqrt{5}x^2 + 1).$$

Similarly

$$f_2(x) = \tilde{c}_2(-\sqrt{5}x^2 + 1).$$

(Note that both c_2 and $\tilde{c}_2$ are arbitrary non-zero numbers which can be chosen independently from each other.)

Exercise. Solve the same problem for (a) $K(x,y) = x - y,\ -1 \le x, y \le 1$, (b) $K(x,y) = \sin x \cos y + \sin y \cos x,\ -\pi \le x, y \le \pi$.