Solutions

Q1. 1) $\sum_{k=1}^{\infty} \frac{2}{10^k} = \frac{2}{10(1-\frac{1}{10})} = \frac{2}{9}$.

$$\sum_{k=1}^{\infty}\left(\frac{1}{10^{2k-1}} + \frac{2}{10^{2k}}\right) = \sum_{k=1}^{\infty} \frac{12}{10^{2k}} = \frac{12}{10^2(1-\frac{1}{10^2})} = \frac{4}{33}.$$

2) $0,2\,(0,35) = 0,2 + \frac{35}{1000(1-\frac{1}{1000})} = 0,2 + \frac{35}{999} = \frac{999+175}{4995} = \frac{1174}{4995}$

Q2. 1) $\sum_{n=1}^{\infty} \frac{1}{(n+1)^{\frac{1}{3}}} = \sum_{n=2}^{\infty} \frac{1}{n^{1/3}} \Rightarrow$ diverging because of (1).

2) $\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}+\frac{1}{n^2}} = 1 \ (= \lim a_n)$.

The serie diverges since the necessary condition for convergence is not satisfied

3) $\lambda = \lim_{n\to\infty} |a_{n+1}/a_n| = \lim_{n\to\infty} \frac{2^{n+1}\, n^8}{2^n (n+1)^8} =$

$2 \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^8} = 2 > 1 \Rightarrow$ diverges.

(ratio test)

4) $\lambda = \lim_{n\to\infty} \frac{(n+1)!\, n^n}{n!\,(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} =$

$\lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1 \Rightarrow$ converges

Here $e = 2.78\ldots$ is the famous limit (we proved the existence of this limit sometime ago).

5) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n(n+2)}} = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

$$0 \le \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n(n+2)}} \le \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow$$

$$\lim_{n\to\infty} a_n = 0.$$

The inequality $a_n \ge a_{n+1}$: $\frac{1}{\sqrt{n(n+2)}} \ge \frac{1}{\sqrt{(n+1)(n+3)}}$ hold because $n(n+2) < (n+1)(n+3)$.

$\Rightarrow$ The series converges (alternating series!).

6) $$\frac{1}{\sqrt{n(n^2+1)}} < \frac{1}{\sqrt{n \cdot n^2}} = \frac{1}{n^{3/2}} \Rightarrow \text{Converges}$$

by comparison test and due to (1) with $s=\frac{3}{2}$.

7) $$\frac{1}{\sqrt{n(n+2)}} > \frac{1}{\sqrt{(n+2)^2}} = \frac{1}{n+2} \Rightarrow \text{diverges}$$

by comparison test and due to $\sum_{n=1}^{\infty} \frac{1}{n+2} =$

$\sum_{n=3}^{\infty} \frac{1}{n}$ (diverges according to (1); $s=1$).

8) $$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1.$$

Hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n-1} \frac{n}{\sqrt{n^2+1}}$ does not exist and the series diverges.

Q3. 1) $2^m a_{2^m} = \dfrac{2^m}{2^m \ln(2^m)} = \dfrac{1}{m \ln(2)}$.

Hence $\displaystyle\sum_{m=1}^{\infty} 2^m a_{2^m} = \frac{1}{\ln(2)} \cdot \sum_{m=1}^{\infty} \frac{1}{m}$.

Applying the same test to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, we obtain: $\displaystyle\sum_{m=1}^{\infty} \frac{2^m}{2^m} = \sum_{m=1}^{\infty} 1 = \infty$.

2) $2^m a_{2^m} = \dfrac{2^m}{2^m (\ln(2^m))^2} = \dfrac{1}{(\ln 2)^2 m^2}$.

$\displaystyle\sum_{m=}^{\infty} \frac{1}{m^2} < \infty$ (converges).

3) $2^m a_{2^m} = \dfrac{2^m \ln 2^m}{2^{2m}} = \dfrac{m \ln 2}{2^m}$.

$\displaystyle\sum_{m=1}^{\infty} \frac{m}{2^m} < \infty$ because of ratio test.

Indeed, $\displaystyle\lim_{m \to \infty} \left(\frac{m+1}{2^{m+1}} \Big/ \frac{m}{2^m} \right) = \lim \frac{m+1}{m \cdot 2} = \frac{1}{2} < 1$.

4) $2^m a_{2^m} = \dfrac{2^m}{2^m \ln(2^m) \ln(\ln 2^m)} = \dfrac{1}{m \ln 2 \cdot \ln(m \ln(2))} =$

$\dfrac{1}{\ln(2) m [\ln(m) + \ln(\ln(2))]}$

Note that $\ln m > \ln(\ln 2) \Leftrightarrow m > \ln 2$. Hence for all $m \geq 1$ $2^m a_{2^m} \geq \dfrac{1}{2\ln(2) \cdot m \ln(m)} \overset{\text{def}}{=} b_m$.

$\displaystyle\sum_{m=1}^{\infty} b_m$ diverges due to 1) above. Hence $\displaystyle\sum_{m=1}^{\infty} a_{2^m} \cdot 2^m = \infty$ (by comparison test). $\Rightarrow$ Diverges.