Solutions to CW 7

Q1. $\lim\limits_{n\to\infty} \frac{1}{n^k}\binom{n}{k} = \lim\limits_{n\to\infty} \frac{1}{n^k}\frac{n(n-1)\dots(n-k+1)}{k!} =$

$\frac{1}{k!}\lim\limits_{n\to\infty}\left(\frac{n}{n}\cdot\frac{n-1}{n}\cdot\ldots\cdot\frac{n-k+1}{n}\right).$

Since $\lim\limits_{n\to} \frac{n-j}{n} = \lim\limits_{n\to\infty}(1-\frac{j}{n}) = 1 - j\lim\limits_{n\to\infty}\frac{1}{n} = 1$

we have:

$$\lim_{n\to\infty}\frac{1}{n^k}\binom{n}{k} = \frac{1}{k!}\lim_{n\to\infty}\frac{n-1}{n}\cdots\lim_{n\to\infty}\frac{n-k+1}{n} = \frac{1}{k!}.$$

Q2. Proof. $a_n \xrightarrow[n\to\infty]{} l \iff \forall\varepsilon>0\ \exists N_1(\varepsilon)$ s.t. $\forall n>N_1(\varepsilon)$

$l-\varepsilon < a_n < l+\varepsilon$ (*)

$b_n \to l \iff \forall\varepsilon>0\ \exists N_2(\varepsilon)$ s.t. $\forall n > N_2(\varepsilon)$

$l-\varepsilon < b_n < l+\varepsilon.$ (**)

For a given $\varepsilon>0$ put $N(\varepsilon) = \max(N_1(\varepsilon), N_2(\varepsilon))$.

Then $\forall n > N(\varepsilon)$ we have:

$l-\varepsilon < a_n \le c_n \le b_n < l+\varepsilon.$ (***)

which implies that $c_n \xrightarrow[n\to\infty]{} l$.

Here (*) and (**) are the conditions of the Lemma. (***) follows from (*), (**), the "sandwich" inequality, and the fact that $n > \max(N_1, N_2)$.

Q3. For a given $\varepsilon > 0$ to have $\left|\frac{1}{n^\alpha} - 0\right| < \varepsilon \Leftrightarrow$

(1) $\frac{1}{n^\alpha} < \varepsilon \Leftrightarrow n^\alpha > \frac{1}{\varepsilon} \Leftrightarrow n > \frac{1}{\varepsilon^{\frac{1}{\alpha}}}$.

Hence, if we choose $N(\varepsilon) = \left\lfloor \frac{1}{\varepsilon^{1/\alpha}} \right\rfloor + 1$, then the inequality (1) holds $\forall\ n > N(\varepsilon)$ $\blacksquare$.

Remark. $\lfloor x \rfloor \doteq$ the largest integer which is $\leq x$.

To prove that $\lim\limits_{n\to\infty} \frac{2n^{2.1} - 3n}{5n^{2.2} + \sqrt{n+1}} = 0$ note that

$$a_n = \frac{2n^{2.1} - 3n}{5n^{2.2} + \sqrt{n+1}} \geq 0 \Leftrightarrow 2n^{2.1} - 3n \geq 0 \Leftrightarrow$$

$$2n^{2.1} \geq 3n \Leftrightarrow 2n^{1.1} \geq 3$$

Since $n^{1.1} \geq n$ if $n \geq 1$, we have that

$$2n \geq 3 \Longrightarrow 2n^{1.1} \geq 3, \text{ or } \quad n \geq 1.5 \Rightarrow a_n \geq 0.$$

In other words: $a_n \geq 0$ if $n \geq 2$.

On the other hand for $n \geq 1$

$$a_n \leq \frac{2n^{2.1}}{5n^{2.2}} \leq \frac{1}{n^{0.1}}$$

Hence for $n \geq 2$ we have

$0 \leq a_n \leq \frac{1}{n^{0.1}}$. This, by the S-Lemma, implies that $\lim\limits_{n\to\infty} a_n = 0$

Q4. Given $x > 1$ we write $x = 1 + a$, where $a > 0$.

By the binomial theorem

$$x^n = (1+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \geq \binom{n}{2} a^2 = \frac{n(n-1)}{2} a^2$$

If $\frac{n(n-1)}{2} a^2 > n$, $\Leftrightarrow$ $\frac{n-1}{2} a^2 > 1 \Leftrightarrow n > 1 + \frac{2}{a^2}$

$(n \geq 1)$

then also $x^n > n$. Hence, if we put

$m = \lfloor 1 + \frac{2}{a^2} \rfloor + 1$, then $\forall n > m$ $x^n > n$.

Put $c_n = \frac{1}{x^n}$. Then for $n > m$, $0 \leq c_n < \frac{1}{n}$.

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} c_n = 0. \blacksquare$$

Q5. As in Q4, if $\ell > k$ and $n > \ell$, then

$$y^{-n} = (1+p)^n \geq \binom{n}{\ell} p^\ell \longrightarrow y^n \leq \frac{1}{\binom{n}{\ell} p^\ell}.$$

Hence

$$n^k y^n \leq \frac{n^k}{\binom{n}{\ell} p^\ell} = \frac{\ell!}{p^\ell} \cdot \frac{n^k}{n(n-1)\dots(n-\ell+1)}$$

To estimate $n(n-1)\dots(n-\ell+1)$, we fix $\ell > k$ and consid $n > 2\ell$. Then $n-j > n-\ell > n-\frac{n}{2} = \frac{n}{2}$ where $0 \leq j \leq \ell$. Hence

$$n(n-1)\dots(n-\ell+1) \geq \frac{n}{2} \cdot \frac{n}{2} \cdot \dots \cdot \frac{n}{2} = \frac{n^\ell}{2^\ell}.$$

But then

$$n^k y^n \leq \frac{\ell! 2^\ell}{p^\ell} \frac{n^k}{n^\ell} = \frac{\ell! 2^\ell}{p^\ell} \cdot \frac{1}{n^{\ell-k}} \xrightarrow[n\to\infty]{} 0.$$

Since, if $c \overset{\text{def}}{=} \frac{\ell! 2^\ell}{p^\ell}$, (by (1), Probl. 3)

$$0 \leq a_n \overset{\text{def}}{=} n^k y^n \leq c \frac{1}{n^{\ell-k}}.$$

it follows from the Sandwich Lemma that

$$\lim_{n\to\infty} a_n = 0. \quad \blacksquare$$