MAS111 CWORK 4

$\boxed{Q1}$ Let (b_n) be a subsequence of (a_n), that is, $b_n = a_{f(n)}$ for some strictly increasing function $f: \mathbb{N} \to \mathbb{N}$. Then $f(n) \geq n$ $n \in \mathbb{N}$ (induction).

Let $\varepsilon > 0$ be given.
Then $\exists m \in \mathbb{N}$ s.t. $\forall n \geq m$ $|a_n - \ell| < \varepsilon$.
For such m, $n \geq m \Rightarrow f(n) \geq m$
$\Rightarrow |a_{f(n)} - \ell| < \varepsilon$, as required.

$\boxed{Q2}$

1) $a_n = 3 - \frac{1}{n}$ $\quad$ $a_n \to 3$ from basic lemmas.

2) $a_n = \frac{5}{2} + \frac{(-1)^n}{4}$ $\quad$ $\min(|a_n - \ell|, |a_{n+1} - \ell|) \geq \frac{1}{2}$ for any $\ell \in \mathbb{R}$, so (a_n) does not converge.

3) $a_n = \frac{(-1)^n}{n}$ $\quad$ $a_{2n} > a_{2n+1}$ $\quad$ $a_{2n+1} < a_{2n+2}$

4) $a_n = n(-1)^n$ $\quad$ $\begin{cases} a_{2n} = 2n \to +\infty \\ a_{2n-1} = -(2n-1) \to -\infty \end{cases}$

(5) $a_n = (-1)^n$ Let $f: \mathbb{N} \to \mathbb{N}$ $n \mapsto 2n$
Then $a_{f(n)} = 1. \to 1$

6) $a_n = n$ $\quad$ For any $f: \mathbb{N} \to \mathbb{N}$ strictly increasing $a_n \leq a_{f(n)}$, hence $(a_{f(n)})$ is unbounded.

7) $a_n = n(1 + (-1)^n)$ $\quad$ $a_{2n} = 2n \to \infty$
$b_n = a_{2n-1} = 0 \to 0$
($f: n \mapsto 2n-1$ is strictly increasing)

8) $a_n := (-1)^n$

$f^{(k)} : \mathbb{N} \to \mathbb{N}$ $\quad n \mapsto \begin{cases} n & n \le k \\ 2n & n > k \end{cases}$ $\quad (k \in \mathbb{N})$

$f^{(k)}$ is strickley increasing.

Let $b_n^{(k)} = a_{f^{(2k)}(n)}$

Then, $\forall k$ $\lim_{n \to \infty} b_n^{(k)} = 1$

The k-sequences are all different.

$b^{(1)} = (-1, 1, 1, 1, \ldots)$

$b^{(2)} = (-1, 1, -1, 1, 1, 1 \ldots)$

$b^{(3)} = (-1, 1, -1, 1, -1, 1, 1, 1 \ldots)$

$\boxed{\text{Q3}}$

1) Let $\varepsilon > 0$ be given.

$$\left|1 - \frac{n+2}{n}\right| = \left|\frac{n-n-2}{n}\right| = \frac{2}{n}$$

Now $\frac{2}{n} < \varepsilon \iff n > \frac{2}{\varepsilon}$

Set $m := \max(1, \lceil 2/\varepsilon \rceil)$

(or $m :=$ smallest integer larger than $\frac{2}{\varepsilon}$)

Then $\forall n \ge m \Rightarrow |1 - a_n| = \frac{2}{n} < \cancel{2} \cdot \frac{\varepsilon}{\cancel{2}} = \varepsilon$.

2) $$\left|\frac{1}{3} - \frac{n^2}{3n^2-1}\right| = \left|\frac{\cancel{3n^2} - 1 - \cancel{3n^2}}{3(3n^2-1)}\right| = \frac{1}{3(3n^2-1)} < \frac{1}{n^2}$$

Let $\varepsilon > 0$ be given.

$\frac{1}{n^2} < \varepsilon \Rightarrow n > \frac{1}{\sqrt{\varepsilon}}$

set $m = \max(1, \lceil \varepsilon^{-1/2} \rceil)$. Then $\forall n \ge m$

$$\left|\frac{1}{3} - a_n\right| = \frac{1}{3(3n-1)} < \frac{1}{n^2} < \varepsilon$$

3) $\left|\frac{2}{5}-\frac{2n^2+n}{5n^2+n+1}\right| = \left|\frac{\cancel{10n^2}+2n+2-\cancel{10n^2}-5n}{5(5n^2+n+1)}\right|$

$\frac{3n-2}{5(5n^2+n+1)} < \frac{3n}{25n^2} = \frac{3}{25n}$

Let $\varepsilon > 0$ be given

$\frac{3}{25n} < \varepsilon \iff n > \frac{3}{25\cdot\varepsilon}$

Set $m := \max(1, \lceil 3/(25\cdot\varepsilon) \rceil)$

Then $\forall n \geq m$

$\left|\frac{2}{5}-a_n\right| < \frac{3}{25n} < \frac{\cancel{3}}{\cancel{25}}\frac{\cancel{25}}{\cancel{3}}\varepsilon = \varepsilon.$

Q4 Basic lemmas

i) $a_n = \ell \Rightarrow a_n \to \ell$

ii) $a_n \to \ell,\ b_n \to \ell' \Rightarrow a_n + b_n \to \ell + \ell'$
$a_n b_n \to \ell \cdot \ell'$

iii) $a_n \to \ell > 0$ and $a_n > 0$ (eventually)
$\Rightarrow \frac{1}{a_n} \to \frac{1}{\ell}$

1) $a_n = \frac{n^2}{5-3n} + \frac{n}{3} = \frac{\cancel{3n^2} + 5n - \cancel{3n^2}}{3(5-3n)} = -5 \cdot \frac{1}{9 - \frac{15}{n}} = b_n \cdot \frac{1}{c_n}$

Now $b_n \xrightarrow{i)} -5$; $c_n = 9 + (-15) \cdot \frac{1}{n}$, and since
$\frac{1}{n} \to 0$ $c_n \xrightarrow{i), ii)} 9 + (-15) \cdot 0 = 9 > 0$.
Finally $c_n > 0$ for $n \geq 2$, hence $\frac{1}{c_n} \xrightarrow{iii} \frac{1}{9}$
and $a_n \xrightarrow{i), ii)} -5 \cdot \frac{1}{9} = -\frac{5}{9}$

2) $a_n = \frac{n}{n + \frac{1}{n + \frac{1}{n}}} = \frac{1}{1 + \frac{1}{n^2+1}} = \frac{n^2+1}{n^2+2} = \left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2}{n^2}\right)^{-1}$

$= b_n \cdot \frac{1}{c_n}$ Now $\frac{1}{n^2} \to 0$ (either from ii) or
as a subsequence of $\frac{1}{n}$) hence $b_n \xrightarrow{ii)} 1 + 0 = 1$.
Likewise $c_n \to 1 + 0 = 1 > 0$ and $c_n > 0$,
so $\frac{1}{c_n} \xrightarrow{iii)} \frac{1}{1} = 1$ and $a_n \xrightarrow{ii)} 1 \cdot 1 = 1$.

3) $b = 1$ $a_n = 1 - \frac{1}{n} = 1 + (-1) \cdot \frac{1}{n} \to 1 + (-1) \cdot 0 = 1$

$\sum_{k=1}^{b} \frac{(-1)^k}{n^k} = 1 \Rightarrow \sum_{k=1}^{b+1} \frac{(-1)^k}{n^k} = 1 + (-1)^{b+1} \frac{1}{n^{b+1}} \to 1 + (-1)^{b+1} \cdot 0 = 1$

So $\sum_{k=0}^{b} \frac{(-1)^k}{n^k} \to 1$ $b \in \mathbb{N}$

$\boxed{Q5}$ We have $(1+x)^n \geq 1+x \qquad x>-1,\ n\in\mathbb{N}$

Thus $\left(1+\frac{1}{n^a}\right)^n \geq 1-n\frac{1}{n^a} = 1-\frac{1}{n^{a-1}}$

$$1-\frac{1}{n^{a-1}} \leq \left(1-\frac{1}{n^a}\right)^n \leq 1^n = 1$$

Since $1-\frac{1}{n^{a-1}} \to 1$, and $1 \to 1$ we have $\left(1-\frac{1}{n^a}\right)^n \to 1$

$\boxed{Q6}$ 1) Let $\varepsilon>0$ be given

Choose $m > \frac{1}{\varepsilon^2}$ (such integer exists from Archimede's principle).

Then $n \geq m \Rightarrow \left|\frac{1}{\sqrt{n}}-0\right| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{m}} < \sqrt{\varepsilon^2} = \varepsilon$.

2) $$a_n = \sqrt{n^a+1}-\sqrt{n^a} = \frac{\left(\sqrt{n^a+1}-\sqrt{n^a}\right)\left(\sqrt{n^a+1}+\sqrt{n^a}\right)}{\sqrt{n^a+1}+\sqrt{n^a}}$$

$$= \frac{\cancel{n^a}+1-\cancel{n^a}}{\sqrt{n^a+1}+\sqrt{n^a}} < \frac{1}{2\sqrt{n^a}} < \frac{1}{(\sqrt{n})^a}$$

Because $\frac{1}{\sqrt{n}} \to 0$ from above, $a_n \to 0$ from basic lemma and sandwich theorem

3) $$a_n = \sqrt{n+\sqrt{n}}-\sqrt{n} = \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} = \frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}+1}$$

Now, $\boxed{\text{for } x \geq 0}$ $(1+x)^2 \geq 1+2x$ (Bernoulli inequality)

$$\Rightarrow \sqrt{1+x} \leq 1+\tfrac{1}{2}x \Rightarrow \frac{1}{\sqrt{1+x}+1} \geq \frac{1}{2+\frac{1}{2}x}$$

Also $\sqrt{1+x} \geq 1 \Rightarrow \frac{1}{\sqrt{1+x}+1} \leq \frac{1}{2}$

Thus

$$\frac{1}{2+\frac{1}{2\sqrt{n}}} \leq a_n \leq \frac{1}{2}$$

Since $\frac{1}{2+\frac{1}{2\sqrt{n}}} \xrightarrow[n\to\infty]{} \frac{1}{2}$, the result follows from sandwich theorem

$\boxed{Q7}$ Assume $(a_n) \xrightarrow[n\to\infty]{} \ell$

Define $f^{(m)} : \mathbb{R} \to \mathbb{R} \quad x \mapsto \sum_{k=0}^{m} c_k x^k$

For $m=0$: $\quad f^{(0)} : x \mapsto c_0$, the constant function.

$$\text{hence } f^{(0)}(a_n) = c_0 = f(\ell).$$

Assume now that, for some $m \geq 0$, and arbitrary $c_0, \dots, c_m$ we have

$$\lim_{n\to\infty} f^{(m)}(a_n) = f(\ell)$$

Then $f^{(m+1)}(x) = x f^{(m)}(x) + c_{m+1}$, so that

$$\lim_{n\to\infty} f^{(m+1)}(a_n) = \lim_{n\to\infty} \left(a_n f^{(m)}(a_n) + c_{m+1} \right)$$

$$= \left(\lim_{n\to\infty} a_n \right) \left(\lim_{n\to\infty} f^{(m)}(a_n) \right) + \lim_{n\to\infty} c_{m+1}$$

$$= \ell \cdot f^{(m)}(\ell) + c_{m+1} = f^{(m+1)}(\ell)$$

where we have used the basic lemmas.