Q1

1. $x\cdot 0 \overset{M1}{=} 0\cdot x \overset{A3}{=} (0+0)\cdot x \overset{D}{=} 0x + 0x \overset{E1}{\Rightarrow} 0x = 0\cdot x + 0\cdot x$
$\overset{E2,A3}{\Rightarrow} \quad 0 = 0\cdot x \overset{E1}{\Rightarrow} \quad 0x = 0$

2. $x(-y) \overset{E3,A4}{=} x(-y) + xy - xy \overset{D}{=} x(-y+y) - xy$
$\overset{A4}{=} x\cdot 0 - xy \overset{1,A3}{=} -xy$
similarly, interchanging x and y, and using M1
$(-x)y = -xy \quad \overset{E1}{\Rightarrow} x(-y) = (-x)y$

3. $(-x)(-y) + x(-y) \overset{D}{=} (-x+x)(-y) \overset{A4}{=} 0y \overset{1}{=} 0.$
hence $(-x)(-y)$ is the additive inverse of $x(-y)$, which is equal to $-xy$, from 2. Thus $(-x)(-y) = -(-xy)$
However $-(-x) \overset{A4}{=} x$, hence $(-x)(-y) = xy$

4. $(x-y)^2 \overset{X}{=} (x-y)(x-y) \overset{D}{=} x(x-y) - y(x-y)$
$\overset{D}{=} x^2 + x(-y) + (-y)x + (-y)(-y) \overset{M1,2,3}{=} x^2 - xy - xy + y^2$
$\overset{M3}{=} x^2 - (xy)\cdot 1 - (xy)\cdot 1 + y^2 \overset{D}{=} x^2 - xy(1+1) + y^2 \overset{?}{=} x^2 - xy\cdot 2 + y^2$
$= x^2 - 2xy + y^2$

5. $x > 0 \overset{E2,A3}{\Rightarrow} x - x > -x \overset{A3}{\Rightarrow} 0 > -x$
$-x < 0 \Rightarrow x - x < x \Rightarrow 0 < x$

6. Now $1 \neq 0$ (M4). If $1 < 0$ and $x > 0 \overset{I4}{\Rightarrow} x\cdot 1 < x\cdot 0$
$\overset{M3,1}{\Rightarrow} x < 0$, a contradiction. Thus $1 > 0$, from I1.

7. $x = 0 \overset{E3}{\Rightarrow} x\cdot x = x\cdot 0 \overset{1}{\Rightarrow} x^2 = 0$
$x > 0 \Rightarrow x\cdot x > x\cdot 0 \Rightarrow x^2 > 0$
$x < 0 \overset{5}{\Rightarrow} -x > 0 \overset{\text{above}}{\Rightarrow} (-x)(-x) > 0 \overset{3}{\Rightarrow} x^2 > 0$
The result follows from I1

8. Let a, b be multiplicative inverses of x. We have

$$a \overset{M3}{=} a \cdot 1 = a \cdot (x \cdot b) \overset{M1}{=} (a \cdot x) \cdot b = 1 \cdot b \overset{M3}{=} b.$$

9. We have $\frac{1}{x} \neq 0$ (from M4, 1).

If $\frac{1}{x} < 0$ then $x \cdot \frac{1}{x} \overset{E3}{<} x \cdot 0 = 0 \overset{M4,\,1}{\Longrightarrow} 1 < 0$, a contradiction

Thus $\frac{1}{x} > 0$. Similarly, $\frac{1}{y} > 0$.

$$x < y \overset{I4}{\Rightarrow} x \cdot \frac{1}{x} \cdot \frac{1}{y} < y \cdot \frac{1}{x} \cdot \frac{1}{y} \overset{M1,\,M3}{\Longrightarrow} \frac{1}{y} < \frac{1}{x}$$

$\boxed{Q2}$

$$x < y \overset{I4}{\Longrightarrow} x^2 < xy$$
$$x < y \overset{I4}{\Longrightarrow} xy < y^2$$
$$\overset{I2}{\Longrightarrow} x^2 < y^2$$

$x^2 < y^2$ and $x, y > 0$. If $y < x$, then, from the above, we would have $y^2 < x^2$, a contradiction. Likewise $x = y \Rightarrow x^2 = xy$ and $xy = y^2 \Rightarrow x^2 = y^2$. So $x < y$, afterall.

Let $x = 0$ and $y = -1$. Then $x \not< y \Rightarrow x^2 < y^2$.

$\boxed{Q3}$ Let $0 < i < n$.

$$B = \binom{n-1}{i} + \binom{n-1}{i-1} = \frac{(n-1)!}{i!\,(n-1-i)!} + \frac{(n-1)!}{(i-1)!\,(n-i)!}$$

$$= \frac{(n-1)!\,(n-\cancel{i}+\cancel{i})}{i!\,(n-i)!} = \frac{n!}{i!\,(n-i)!} = \binom{n}{i}$$

If $i = 0$, then $B = \binom{n-1}{0} = 1 = \binom{n}{0}$

If $i = n$, then $B = \binom{n-1}{n-1} = 1 = \binom{n}{n}$

If $i < 0$ or $i > n$, then $B = 0 = \binom{n}{i}$

Q4 $n=1$ $\quad LHS = (x+y)^1 = x+y.$

$$RHS = \binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0 = x+y.$$

Assume $LHS(n) = RHS(n)$ for some $n \geq 1$

$$LHS(n+1) = (x+y)^n(x+y) = \sum_{i=0}^{n}\binom{n}{i}x^{i+1}y^{n-i} + \sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i+1}$$

$$= \sum_{i=1}^{n+1}\binom{n}{i-1}x^i y^{n-i+1} + \sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i+1}$$

$$= \sum_{i=1}^{n}\left[\binom{n}{i-1}+\binom{n}{i}\right]x^i y^{n-i+1} + \binom{n}{n}x^{n+1} + \binom{n}{0}y^{n+1}$$

$$= \sum_{i=1}^{n}\binom{n+1}{i}x^i y^{(n+1)-i} + \binom{n+1}{n+1}x^{n+1} + \binom{n+1}{0}y^{n+1} = RHS(n+1).$$

Q5 $LHS(1) = 1+x = RHS(1)$. Suppose $LHS(n) \geq RHS(n)$ for some $n \geq 1$. Then, since $x > -1 \Rightarrow 1+x > 0$,
$LHS(n+1) = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1+(n+1)x + nx^2$
$\geq 1+(n+1)x = RHS(n+1).$

Q6 For $n=1$, we have $LHS(1) = 1^3 = 1^2 = RHS(1)$.
Assume $LHS(n) = RHS(n)$, for some $n \geq 1$. Then

$$LHS(n+1) = \sum_{k=1}^{n}k^3 + (n+1)^3 = \left(\sum_{k=1}^{n}k\right)^2 + (n+1)^3 = \frac{n^2(n+1)^2}{2^2} + (n+1)^3$$

$$= \frac{(n+1)^2[n^2+4(n+1)]}{2^2} = \frac{(n+1)^2(n+2)^2}{2^2} = RHS(n+1).$$