MAS/111 Convergence and Continuity: Coursework 7

DEADLINE: Thursday of week 10, at 11:00 am.

Problem 1. Compute the following limit: $\lim_{n\to\infty} \frac{1}{n^k}\binom{n}{k}$.

Problem 2. Prove the Sandwich Lemma.

Lemma. *Let (a_n), (b_n), and (c_n) be three sequences of real numbers such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \ell$, and (c_n) is "sandwiched" by the two other sequences, that is $\forall\ n \geq 1$ the inequality $a_n \leq c_n \leq b_n$ holds true. Then $\lim_{n\to\infty} c_n = \ell$.*

Problem 3. Prove from the first principles that for any $\alpha > 0$

$$\lim_{n\to\infty} \frac{1}{n^\alpha} = 0 \tag{1}$$

Using (1) and the Sandwich Lemma prove that

$$\lim_{n\to\infty} \frac{2n^{2.1} - 3n}{5n^{2.2} + \sqrt{n+1}} = 0.$$

Problem 4. Prove that for any $x > 1$ there is $m \in \mathbb{N}$ such that for any $n > m$ the inequality $x^n > n$ holds.

Using this fact and the Sandwich Lemma prove that

$$\lim_{n\to\infty} \frac{1}{x^n} = 0. \tag{2}$$

Problem 5. Let $0 < y < 1$ and $k \geq 0$. Prove that

$$\lim_{n\to\infty} n^k y^n = 0 \text{ an in particular } \lim_{n\to\infty} y^n = 0$$

[Let $y^{-1} = 1 + p$, $p > 0$. Use the binomial theorem to show that $(1+p)^n \geq \binom{n}{l} p^l$, where $l \in \mathbb{N}$, $l \leq n$. Note also that $n^k < n^l$ if $k < l$. These inequalities allow you to bound above the $n^k y^n$ and to use the Sandwich Lemma.]