MAS/111 Convergence and Continuity: Coursework 6

DEADLINE: Thursday of week 9, at 11:00 am.

Problem 1. Find the supremum and infimum of each of the following sets

1) $\left\{\dfrac{n+1}{n+3} \,:\, n \in \mathbb{N}\right\}$

2) $\left\{\dfrac{1}{1+x^2} \,:\, x \in \mathbb{R}\right\}$

3) $\left\{\dfrac{1}{2^n} + \dfrac{1}{3^m} \,:\, m, n \in \mathbb{N}\right\}$

4) $\{x + y \,:\, x, y \in \mathbb{R},\, x^2 + 2x - 3 < 0,\, y^2 < 9\}$

Problem 2. In each case below, state whether the function f
(i) is bounded above, and if so whether it attains its upper bound
(ii) is bounded below, and if so whether it attains its lower bound

1) $f : (0, 1] \to \mathbb{R}$ $\qquad x \mapsto \dfrac{1}{x}$

2) $f : \mathbb{Z} \to \mathbb{R}$ $\qquad x \mapsto \begin{cases} 0 & \text{if } x^2 \le 3 \\ 1 & \text{if } x^2 > 3 \end{cases}$

3) $f : \mathbb{R} \to \mathbb{R}$ $\qquad x \mapsto \dfrac{x^2 - 4}{x^2 + 1}$

4) $f : \mathbb{R} \setminus \{1, -1\} \to \mathbb{R}$ $\qquad x \mapsto \dfrac{x^2 + 4}{x^2 - 1}$

[In 4), plot the graph of f near $x = 1$.]

Problem 3. Let r be a rational number. Prove that the set

$$X = \{x \in \mathbb{Q} \,:\, x < r\}$$

does not have a greatest element.

[Use contradiction.]

Problem 4. Show that a set of real numbers cannot have two different greatest lower bounds.

[Definition!]

Problem 5. Use the Intermediate Value Theorem to prove that every positive real number has a square root.

[Find a continuous function which vanishes at the desired root.]

Problem 6. Let $f : [0,1] \to [0,1]$ be a continuous function. Use the Intermediate Value Theorem to prove that f has a *fixed point;* that is, prove that there exists $c \in [0,1]$ such that $f(c) = c$.

Hint: consider the function $g(x){=}f(x) - x$. What can you say about the sign of $g(0)$? $g(1)$? Does this function have a root in $[0,1]$?