MAS/111 Convergence and Continuity: Coursework 4

DEADLINE: Thursday of week 6, at 11:00 am.

Problem 1. Prove that if a real sequence converges to a limit, then every subsequence also converges to the same limit.

Problem 2. Give examples of real sequences (a_n), such that

1) $a_n \in [2, 3)$ and (a_n) converges to 3.

2) $a_n \in (2, 3)$ and (a_n) does not converge.

3) (a_n) converges and is neither increasing nor decreasing.

4) (a_n) is not bounded above, and not bounded below.

*Remember:*We say that (a_n) is bounded below if there is a $c \in \mathbb{R}$ such that $a_n \geq c$ for all n; (a_n) is bounded above if there is a $d \in \mathbb{R}$ such that $a_n \leq c$ for all n.)

5) (a_n) does not converge, but it has a convergent subsequence.

Briefly justify your answer in each case.

Problem 3. Prove from first principles

$$1) \quad \lim_{n\to\infty} \frac{n+2}{n} = 1$$

$$2) \quad \lim_{n\to\infty} \frac{n^2}{3n^2-1} = \frac{1}{3}$$

$$3) \quad \lim_{n\to\infty} \frac{2n^2+n}{5n^2+n+1} = \frac{2}{5}$$

Problem 4. Using the basic lemmas (chapter 3), compute $\lim_{n\to\infty} a_n$ for the following values of a_n

$$1)\quad a_n = \frac{n^2}{5-3n} + \frac{n}{3} \qquad 2)\quad a_n = \frac{n}{n + \cfrac{1}{n + \cfrac{1}{n}}} \qquad 3)\quad a_n = \sum_{k=0}^{b} \frac{(-1)^k}{n^k} \quad b \in \mathbb{N}$$

[Explain which lemmas are used at each stage of the proofs. In 3), use induction on b.]

Problem 5. Prove that

$$\lim_{n\to\infty} \left(1 - \frac{1}{n^a}\right)^n = 1 \qquad a \in \mathbb{N},\ a > 1.$$

[Use Bernoulli inequality[1], the fact that $\lim_{n\to\infty} \frac{1}{n^b} = 0$ for any $b > 0$, and the definition of the limit.]

Problem 6. Prove that

$$1)\quad \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

hence compute

$$2)\quad \lim_{n\to\infty} \sqrt{n^a+1} - \sqrt{n^a} \qquad 3)\quad \lim_{n\to\infty} \sqrt{n + \sqrt{n}} - \sqrt{n} \qquad a \in \mathbb{N}.$$

[Transfer square roots at denominator: $\sqrt{A} - \sqrt{B} = \frac{A-B}{\sqrt{A}+\sqrt{B}}$.]

Problem 7. Let m be a non-negative integer, let $c_0, \ldots, c_m$ be real numbers, and let

$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto \sum_{k=0}^{m} c_k x^k$$

Prove that if (a_n) is a sequence of real numbers converging to l, then the sequence $(f(a_n))$ converges to $f(l)$.

[Use induction].

[1] problem 5, cwork 1