MAS/111 Convergence and Continuity: Coursework 3

DEADLINE: Thursday of week 5, at 11:00 am.

Problem 1. Prove that, for all $n \in \mathbb{N}$

1) $\dfrac{1}{2} \leq \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{2n} < 1$

2) $2^{n-1} \leq n! \leq n^{n-1}$.

Problem 2. Prove that $\exists m \in \mathbb{N}$ s.t. $\forall n \geq m \quad 2^n \geq n^2$.

Problem 3. Prove that, for all $n \in \mathbb{N}$

1) $$\sum_{k=1}^{n} k! \cdot k = (n+1)! - 1$$

2) $$\sum_{k=1}^{n} \frac{k^2}{2^k} = 6 - \frac{1}{2^n}(n^2 + 4n + 6)$$

3) $$\sum_{k=2}^{n} \frac{1}{k^2 - 1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$$

4) $$\sum_{k=1}^{n} \frac{1}{[\,p + (k-1)q\,](p + kq)} = \frac{n}{p(p+nq)} \qquad p, q \in \mathbb{R} \quad p, q > 0.$$

[Use induction. The identity 4) will be useful in problem 4.]

Problem 4. Consider the statement

$$\exists m > 0 \text{ s.t. } \forall n \geq m \quad \mathcal{P}(n) \tag{1}$$

for the following expressions $\mathcal{P}(n)$

1) $$\left|\frac{1}{3} - \frac{n}{3n-1}\right| < 10^{-2}$$

2) $$\left|\frac{1}{3} - \frac{n}{3n^2-1}\right| < 10^{-2}$$

3) $$\left|\sum_{r=1}^{n} \frac{1}{(5r-4)(5r+1)} - \frac{1}{5}\right| < \frac{1}{1000}.$$

In each case, if the statement (1) is true, prove it by providing an appropriate integer m. If (1) is false, state explicitly its negation, and then prove it.

Problem 5. Let (x_n) be the sequence defined recursively as

$$x_1 = 1 \qquad x_{n+1} = \frac{x_n^2}{2x_n + 2} \qquad n \in \mathbb{N}.$$

1) Prove that

$$0 < x_{n+1} < x_n$$

2) Prove that

$$\exists m \in \mathbb{N} \text{ s.t. } \forall n \geq m \qquad x_n < \frac{1}{10^{300}}.$$