# 7 The Simplex Method

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Further let B be a *basis*, i.e., a set $B \subseteq \{1, \ldots, n\}$ with $|B| = m$, corresponding to a choice of $m$ non-zero variables. Let $x \in \mathbb{R}^n$ such that $Ax = b$. Then we have

$$A_B x_B + A_N x_N = b,$$

where $A_B \in \mathbb{R}^{m \times m}$ and $A_N \in \mathbb{R}^{m \times (n-m)}$ respectively consist of the columns of $A$ indexed by B and those not indexed by B, and $x_B$ and $x_N$ respectively consist of the rows of $x$ indexed by B and those not indexed by B. Moreover, if $x$ is a basic solution, then there is a basis B such that $x_N = 0$ and $A_B x_B = b$, and if $x$ is a basic feasible solution, there is a basis B such that $x_N = 0$, $A_B x_B = b$, and $x_B \geqslant 0$.

## 7.1 The Simplex Tableau

For every $x$ with $Ax = b$ and every basis B, we have that $x_B = A_B^{-1}(b - A_N x_N)$, and thus

$$
\begin{aligned}
f(x) = c^\mathsf{T} x &= c_B^\mathsf{T} x_B + c_N^\mathsf{T} x_N \\
&= c_B^\mathsf{T} A_B^{-1}(b - A_N x_N) + c_N^\mathsf{T} x_N \\
&= c_B^\mathsf{T} A_B^{-1} b + (c_N^\mathsf{T} - c_B^\mathsf{T} A_B^{-1} A_N) x_N
\end{aligned}
$$

Suppose that we want to maximize $c^\mathsf{T} x$ and find that

$$c_N^\mathsf{T} - c_B^\mathsf{T} A_B^{-1} A_N \leqslant 0 \quad \text{and} \quad A_B^{-1} b \geqslant 0. \tag{7.1}$$

Then, for any feasible $x \in \mathbb{R}^n$, it holds that $x_N \geqslant 0$ and therefore $f(x) \leqslant c_B^\mathsf{T} A_B^{-1} b$. The basic solution $x^*$ with $x_B^* = A_B^{-1} b$ and $x_N^* = 0$, on the other hand, is feasible and satisfies $f(x^*) = c_B^\mathsf{T} A_B^{-1} b$. It must therefore be optimal.

If alternatively $(c_N^\mathsf{T} - c_B^\mathsf{T} A_B^{-1} A_N)_i > 0$ for some $i$, then we can increase the value of the objective by increasing $(x_N)_i$. Either this can be done indefinitely, which means that the maximum is unbounded, or the constraints force some of the variables in the basis to become smaller and we have to stop when the first one reaches zero. In that case we have found a new BFS with a larger value and can repeat the process.

Assuming that the LP is feasible and has a bounded optimal solution, there exists a basis $B^*$ for which (7.1) is satisfied. The basic idea behind the simplex method is to start from an initial BFS and then move from basis to basis until $B^*$ is found. The information required for this procedure can conveniently be represented by the so-called *simplex tableau*. For a given basis B, it takes the following form:[1]

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[1] The columns of the tableau have been permuted such that those corresponding to the basis appear on the left. This has been done just for convenience: in practice we will always be able to identify the columns corresponding to the basis by the embedded identity matrix.

$$
\begin{array}{c}
\overbrace{\phantom{AAAAAAAA}}^{m} \quad \overbrace{\phantom{AAAAAAAA}}^{n-m} \quad \overbrace{\phantom{AAA}}^{1} \\
\end{array}
$$

|   | B | N |   |
|---|---|---|---|
| $m$ | $A_B^{-1}A_B = I$ | $A_B^{-1}A_N$ | $A_B^{-1}b$ |
| $1$ | $c_B^\mathsf{T} - c_B^\mathsf{T}A_B^{-1}A_B = 0$ | $c_N^\mathsf{T} - c_B^\mathsf{T}A_B^{-1}A_N$ | $-c_B^\mathsf{T}A_B^{-1}b$ |

The first $m$ rows consist of the matrix $A$ and the column vector $b$, multiplied by the inverse of $A_B$. It is worth pointing out that for any basis B, the LP with constraints $A_B^{-1}Ax = A_B^{-1}b$ is equivalent to the one with constraints $Ax = b$. The first $n$ columns of the last row are equal to $c^\mathsf{T} - \lambda^\mathsf{T}A$ for $\lambda^\mathsf{T} = c_B^\mathsf{T}A_B^{-1}$. The vector $\lambda$ can be interpreted as a solution, not necessarily feasible, to the dual problem. In the last column of the last row we finally have the value $-f(x)$, where $x$ is the BFS given by $x_B = A_B^{-1}b$ and $x_N = 0$.

 We will see later that the simplex method always maintains feasibility of this solution $x$. As a consequence it also maintains complementary slackness for $x$ and $\lambda^\mathsf{T} = c_B^\mathsf{T}A_B^{-1}$: since we work with an LP in standard form, $\lambda^\mathsf{T}(Ax - b) = 0$ follows automatically from the feasibility condition, $Ax = b$; the condition $(c^\mathsf{T} - \lambda^\mathsf{T}A)x = 0$ holds because $x_N = 0$ and $c_B^\mathsf{T} - \lambda^\mathsf{T}A_B = c_B^\mathsf{T} - c_B^\mathsf{T}A_B^{-1}A_B = 0$. What it then means for (7.1) to become satisfied is that $c^\mathsf{T} - \lambda^\mathsf{T}A \leqslant 0$, i.e., that $\lambda$ is a feasible solution for the dual. Optimality of $x$ is thus actually a consequence of Theorem 6.3.

## 7.2   Using The Tableau

Consider a tableau of the following form, where the basis can be identified by the identity matrix embedded in $(a_{ij})$:

|   |   |
|---|---|
| $(a_{ij})$ | $a_{i0}$ |
| $a_{0j}$ | $a_{00}$ |

The simplex method then takes the following steps:

1. Find an initial BFS with basis B.

2. Check whether $a_{0j} \leqslant 0$ for every $j$. If yes, the current solution is optimal, so stop.

3. Choose $j$ such that $a_{0j} > 0$, and choose $i \in \{i' : a_{i'j} > 0\}$ to minimize $a_{i0}/a_{ij}$. If $a_{ij} \leqslant 0$ for all $i$, then the problem is unbounded, so stop. If multiple rows minimize $a_{i0}/a_{ij}$, the problem is degenerate.

4. Update the tableau by multiplying row $i$ by $1/a_{ij}$ and adding a $-(a_{kj}/a_{ij})$ multiple of row $i$ to each row $k \neq i$. Then return to Step 2.

We will now describe the different steps of the simplex method in more detail and illustrate them using the LP of Example 1.1.

## Finding an initial BFS

Finding an initial BFS is very easy when the constraints are of the form $Ax \leqslant b$ for $b \geqslant 0$. We can then add a vector $z$ of slack variables and write the constraints as $Ax + z = b$, $z \geqslant 0$ and get a BFS by setting $x = 0$ and $z = b$. This can alternatively be thought of as extending $x$ to $(x, z)$ and setting $(x_B, x_N) = (z, x) = (b, 0)$. We then have $A_B^{-1} = I$ and $c_B = 0$, and the entries in the tableau become $A_N$ and $c_N^\mathsf{T}$ for the variables $x_1$ and $x_2$ that are not in the basis, and $b$ and $0$ in the last column. For the LP of Example 1.1 we obtain the following tableau, where rows and columns have been labeled with the names of the corresponding variables:

|       | $x_1$ | $x_2$ | $z_1$ | $z_2$ | $a_{i0}$ |
|-------|-------|-------|-------|-------|----------|
| $z_1$ | 1     | 2     | 1     | 0     | 6        |
| $z_2$ | 1     | $-1$  | 0     | 1     | 3        |
| $a_{0j}$ | 1  | 1     | 0     | 0     | 0        |

If the constraints do not have this convenient form, finding an initial BFS requires more work. We will discuss this case in the next lecture.

## Choosing a pivot column

If $a_{0j} \leqslant 0$ for all $j \geqslant 1$, the current solution is optimal. Otherwise we can choose a column $j$ such that $a_{0j} > 0$ as the pivot column and let the corresponding variable enter the basis. If multiple candidate columns exist, choosing any one of them will work, but we could for example break ties toward the one that maximizes $a_{0j}$ or the one with the smallest index. The candidate variables in our example are $x_1$ and $x_2$, so let us choose $x_1$.

## Choosing the pivot row

If $a_{ij} \leqslant 0$ for all $i$, then the problem is unbounded and the objective can be increased by an arbitrary amount. Otherwise we choose a row $i \in \{i' : a_{i'j} > 0\}$ that minimizes $a_{i0}/a_{ij}$. This row is called the pivot row, and $a_{ij}$ is called the pivot. If multiple rows minimize $a_{i0}/a_{ij}$, the problem has a degenerate BFS. In our example there is a unique choice, corresponding to variable $z_2$.

## Pivoting

The purpose of the pivoting step is to get the tableau into the appropriate form for the new BFS. For this, we multiply row $i$ by $1/a_{ij}$ and add a $-(a_{kj}/a_{ij})$ multiple of row $i$

to each row $k \neq i$, including the last one. Our choice of the pivot row as a row that minimizes $a_{i0}/a_{ij}$ turns out to be crucial, as it guarantees that the solution remains feasible after pivoting. In our example, we need to subtract the second row from both the first and the last row, after which the tableau looks as follows:

|       | $x_1$ | $x_2$ | $z_1$ | $z_2$ | $a_{i0}$ |
|-------|-------|-------|-------|-------|----------|
| $z_1$ | 0     | 3     | 1     | $-1$  | 3        |
| $x_1$ | 1     | $-1$  | 0     | 1     | 3        |
| $a_{0j}$ | 0  | 2     | 0     | $-1$  | $-3$     |

Note that the second row now corresponds to variable $x_1$, which has replaced $z_2$ in the basis.

We are now ready to choose a new pivot column. In our example, one further iteration yields the following tableau:

|       | $x_1$ | $x_2$ | $z_1$ | $z_2$ | $a_{i0}$ |
|-------|-------|-------|-------|-------|----------|
| $x_2$ | 0     | 1     | $\frac{1}{3}$  | $-\frac{1}{3}$  | 1 |
| $x_1$ | 1     | 0     | $\frac{1}{3}$  | $\frac{2}{3}$   | 4 |
| $a_{0j}$ | 0  | 0     | $-\frac{2}{3}$ | $-\frac{1}{3}$  | $-5$ |

This corresponds to the BFS where $x_1 = 4$, $x_2 = 1$, and $z_1 = z_2 = 0$, with an objective of $-5$. All entries in the last row are non-positive, so this solution is optimal.