4 Conditions for Strong Duality

While we have already solved a few optimization problems using the method of Lagrange multipliers, it was not clear a priori whether each individual problem satisfied strong duality and whether our attempt to solve it would ultimately be successful. Our goal in this lecture will be to identify general conditions that guarantee strong duality, and classes of problems that satisfy these conditions.

4.1 Supporting Hyperplanes and Convexity

To this end, we again consider the function ϕ that describes how the optimal value behaves as we vary the right-hand side of the constraints. Fix a particular $b \in \mathbb{R}^m$ and consider $\phi(c)$ as a function of $c \in \mathbb{R}^m$. Further consider the hyperplane given by $\alpha : \mathbb{R}^m \to \mathbb{R}$ with

$$\alpha(c) = \beta + \lambda^T(c - b).$$

This hyperplane has intercept β at b and slope λ. We can now try to find $\phi(b)$ as follows:

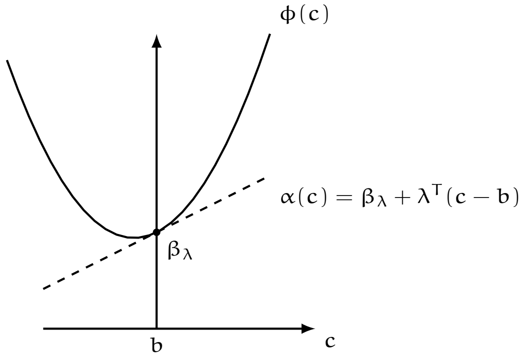
1. For each λ, find $\beta_\lambda = \sup\{\beta : \beta + \lambda^T(c - b) \leqslant \phi(c) \text{ for all } c \in \mathbb{R}^m\}$.
2. Choose λ to maximize β_λ.

This approach is illustrated in Figure 4.1. We always have that $\beta_\lambda \leqslant \phi(b)$. In the situation on the left of Figure 4.1, this condition holds with equality because there is a tangent to ϕ at b. In fact,

$$\begin{aligned}
g(\lambda) &= \inf_{x \in X} L(x, \lambda) \\
&= \inf_{x \in X} \big(f(x) - \lambda^T(h(x) - b)\big) \\
&= \inf_{c \in \mathbb{R}^m} \inf_{x \in X(c)} \big(f(x) - \lambda^T(h(x) - b)\big) \\
&= \inf_{c \in \mathbb{R}^m} \big(\phi(c) - \lambda^T(c - b)\big) \\
&= \sup\big\{\beta : \beta + \lambda^T(c - b) \leqslant \phi(c) \text{ for all } c \in \mathbb{R}^m\big\} \\
&= \beta_\lambda
\end{aligned}$$

We again see the weak duality result as $\max_\lambda \beta_\lambda \leqslant \phi(b)$, but we also obtain a condition for strong duality. Call $\alpha : \mathbb{R}^m \to \mathbb{R}$ a *supporting hyperplane* to ϕ at b if $\alpha(c) = \phi(b) + \lambda^T(c - b)$ and $\phi(c) \geqslant \phi(b) + \lambda^T(c - b)$ for all $c \in \mathbb{R}^m$.

THEOREM 4.1. *Problem* (1.1) *satisfies strong duality if and only if there exists a (non-vertical) supporting hyperplane to* ϕ *at* b.



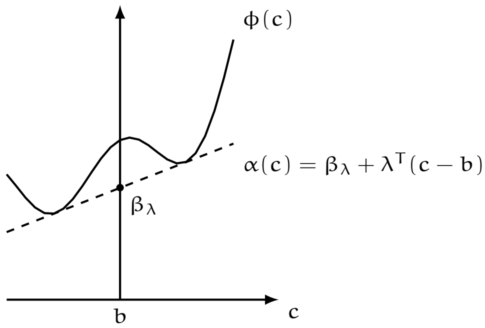


Figure 4.1: Geometric interpretation of the dual with optimal value $g(\lambda) = \beta_\lambda$. In the situation on the left strong duality holds, and $\beta_\lambda = \phi(b)$. In the situation on the right, strong duality does not hold, and $\beta_\lambda < \phi(b)$.

Proof. Suppose there exists a (non-vertical) supporting hyperplane to ϕ at b. This means that there exists $\lambda \in \mathbb{R}^m$ such that for all $c \in \mathbb{R}^m$,

$$\phi(b) + \lambda^T(c - b) \leqslant \phi(c).$$

This implies that

$$\begin{aligned} \phi(b) &\leqslant \inf_{c \in \mathbb{R}^m} \left(\phi(c) - \lambda^T(c - b)\right) \\ &= \inf_{c \in \mathbb{R}^m} \inf_{x \in X(c)} \left(f(x) - \lambda^T(h(x) - b)\right) \\ &= \inf_{x \in X} L(x, \lambda) \\ &= g(\lambda). \end{aligned}$$

On the other hand, $\phi(b) \geqslant g(\lambda)$ by Theorem 3.2, so $\phi(b) = g(\lambda)$ and strong duality holds.

Now suppose that the problem satisfies strong duality. Then there exists $\lambda \in \mathbb{R}^m$ such that for all $c \in \mathbb{R}^m$

$$\begin{aligned} \phi(b) = g(\lambda) &= \inf_{x \in X} L(x, \lambda) \\ &\leqslant \inf_{x \in X(c)} L(x, \lambda) \\ &= \inf_{x \in X(c)} (f(x) - \lambda^T(h(x) - b)) \\ &= \phi(c) - \lambda^T(c - b), \end{aligned}$$

and thus

$$\phi(b) + \lambda^T(c - b) \leqslant \phi(c).$$

This describes a (non-vertical) supporting hyperplane to ϕ at b. □

A sufficient condition for the existence of a supporting hyperplane is provided by the following basic result, which we state without proof.

THEOREM 4.2 (Supporting Hyperplane Theorem). *Suppose that* $\phi : \mathbb{R}^m \to \mathbb{R}$ *is convex and* $b \in \mathbb{R}^m$ *lies in the interior of the set of points where* ϕ *is finite. Then there exists a (non-vertical) supporting hyperplane to* ϕ *at* b.

4.2 A Sufficient Condition for Convexity

We now know that convexity of ϕ guarantees strong duality for every constraint vector b, but it is not clear how to recognize optimization problems that have this property. The following result identifies a sufficient condition.

THEOREM 4.3. *Consider the problem to*

$$\begin{aligned} \textit{minimize} \quad & f(x) \\ \textit{subject to} \quad & h(x) \leqslant b \\ & x \in X, \end{aligned}$$

and let ϕ *be given by* $\phi(b) = \inf_{x \in X(b)} f(x)$. *Then,* ϕ *is convex if* X, f, *and* h *are convex.*

Proof. Consider $b_1, b_2 \in \mathbb{R}^m$ such that $\phi(b_1)$ and $\phi(b_2)$ are defined, and let $\delta \in [0, 1]$ and $b = \delta b_1 + (1 - \delta) b_2$. Further consider $x_1 \in X(b_1)$, $x_2 \in X(b_2)$, and let $x = \delta x_1 + (1 - \delta) x_2$. Then convexity of X implies that $x \in X$, and convexity of h that

$$\begin{aligned} h(x) &= h(\delta x_1 + (1 - \delta) x_2) \\ &\leqslant \delta h(x_1) + (1 - \delta) h(x_2) \\ &\leqslant \delta b_1 + (1 - \delta) b_2 \\ &= b. \end{aligned}$$

Thus $x \in X(b)$, and by convexity of f,

$$\phi(b) \leqslant f(x) = f(\delta x_1 + (1 - \delta) x_2) \leqslant \delta f(x_1) + (1 - \delta) f(x_2).$$

This holds for all $x_1 \in X(b_1)$ and $x_2 \in X(b_2)$, so taking infima on the right hand side yields

$$\phi(b) \leqslant \delta \phi(b_1) + (1 - \delta) \phi(b_2).$$

□

Note that an equality constraint $h(x) = b$ is equivalent to the pair of constraints $h(x) \leqslant b$ and $-h(x) \leqslant -b$. In this case, the above result requires that X, f, h, and $-h$ are all convex, which in particular requires that h is linear. Indeed, in the case with equality constraints, convexity of f and h does not suffice for convexity of ϕ. To see this, consider the problem to

$$\text{minimize } f(x) = x^2 \text{ subject to } h(x) = x^3 = b$$

for some $b > 0$. Then $\phi(b) = b^{2/3}$, which is not convex. The Lagrangian is $L(x, \lambda) = x^2 - \lambda(x^3 - b) = (x^2 - \lambda x^3) + \lambda b$, and has a finite minimum if and only if $\lambda = 0$. The dual thus has an optimal value of 0, which is strictly smaller than $\phi(b)$ if $b > 0$.

Linear programs satisfy the conditions, both for equality and inequality constraints. We thus have the following.

THEOREM 4.4. *If a linear program is feasible and bounded, it satisfies strong duality.*