2 The Method of Lagrange Multipliers

A well-known method for solving constrained optimization problems is the method of Lagrange multipliers. The idea behind this method is to reduce constrained optimization to unconstrained optimization, and to take the (functional) constraints into account by augmenting the objective function with a weighted sum of them. To this end, define the *Lagrangian* associated with (1.1) as

$$L(x, \lambda) = f(x) - \lambda^T(h(x) - b), \tag{2.1}$$

where $\lambda \in \mathbb{R}^m$ is a vector of *Lagrange multipliers*.

2.1 Lagrangian Sufficiency

The following result provides a condition under which minimizing the Lagrangian, subject only to the regional constraints, yields a solution to the original constrained problem. The result is easy to prove, yet extremely useful in practice.

THEOREM 2.1 (Lagrangian Sufficiency Theorem). *Let* $x \in X$ *and* $\lambda \in \mathbb{R}^m$ *such that* $L(x, \lambda) = \inf_{x' \in X} L(x', \lambda)$ *and* $h(x) = b$*. Then* x *is an optimal solution of* (1.1).

Proof. We have that

$$\begin{aligned} \min_{x' \in X(b)} f(x') &= \min_{x' \in X(b)} [f(x') - \lambda^T(h(x') - b)] \\ &\geqslant \min_{x' \in X} [f(x') - \lambda^T(h(x') - b)] \\ &= f(x) - \lambda^T(h(x) - b) = f(x). \end{aligned}$$

Equality in the first line holds because $h(x') - b = 0$ when $x' \in X(b)$. The inequality on the second line holds because the minimum is taken over a larger set. In the third line we finally use that x minimizes L and that $h(x) = b$. □

Two remarks are in order. First, a vector λ of Lagrange multipliers satisfying the conditions of the theorem is not guaranteed to exist in general, but it does exist for a large class of problems. Second, the theorem appears to be useful mainly for showing that a given solution x is optimal. In certain cases, however, it can also be used to find an optimal solution. Our general strategy in these cases will be to minimize $L(x, \lambda)$ for all values of λ, in order to obtain a minimizer $x^*(\lambda)$ that depends on λ, and then find λ^* such that $x^*(\lambda^*)$ satisfies the constraints.

2.2 Using Lagrangian Sufficiency

We begin by applying Theorem 2.1 to a concrete example.

EXAMPLE 2.2. Assume that we want to

$$\begin{aligned} \text{minimize} \quad & x_1 - x_2 - 2x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 5 \\ & x_1^2 + x_2^2 = 4. \end{aligned}$$

The Lagrangian of this problem is

$$\begin{aligned} L(x, \lambda) &= x_1 - x_2 - 2x_3 - \lambda_1(x_1 + x_2 + x_3 - 5) - \lambda_2(x_1^2 + x_2^2 - 4) \\ &= \left((1 - \lambda_1)x_1 - \lambda_2 x_1^2\right) + \left((-1 - \lambda_1)x_2 - \lambda_2 x_2^2\right) + \left((-2 - \lambda_1)x_3\right) + 5\lambda_1 + 4\lambda_2. \end{aligned}$$

For a given value of λ, we can minimize $L(x, \lambda)$ by independently minimizing the terms in x_1, x_2, and x_3, and we will only be interested in values of λ for which the minimum is finite.

The term $(-2 - \lambda_1)x_3$ does not have a finite minimum unless $\lambda_1 = -2$. The terms in x_1 and x_2 then have a finite minimum only if $\lambda_2 < 0$, in which case an optimum occurs when

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 1 - \lambda_1 - 2\lambda_2 x_1 = 3 - 2\lambda_2 x_1 = 0 \quad \text{and} \\ \frac{\partial L}{\partial x_2} &= -1 - \lambda_1 - 2\lambda_2 x_2 = 1 - 2\lambda_2 x_2 = 0, \end{aligned}$$

i.e., when $x_1 = 3/(2\lambda_2)$ and $x_2 = 1/(2\lambda_2)$. The optimum is indeed a minimum, because

$$\mathcal{H}L = \begin{pmatrix} \frac{\partial^2 L}{\partial x_1 \partial x_1} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2 \partial x_2} \end{pmatrix} = \begin{pmatrix} -2\lambda_2 & 0 \\ 0 & -2\lambda_2 \end{pmatrix},$$

is positive semidefinite when $\lambda_2 < 0$.

Let Y be the set of values of λ such that $L(x, \lambda)$ has a finite minimum, i.e.,

$$Y = \{\lambda \in \mathbb{R}^2 : \lambda_1 = -2, \lambda_2 < 0\}.$$

For every $\lambda \in Y$, the unique optimum of $L(x, \lambda)$ occurs at $x^*(\lambda) = (3/(2\lambda_2), 1/(2\lambda_2), x_3)^T$, and we need to find $\lambda \in Y$ such that $x^*(\lambda)$ is feasible to be able to apply Theorem 2.1. Therefore,

$$x_1^2 + x_2^2 = \frac{9}{4\lambda_2^2} + \frac{1}{4\lambda_2^2} = 4$$

and thus $\lambda_2 = -\sqrt{5/8}$. We can now use Theorem 2.1 to conclude that the minimization problem has an optimal solution at $x_1 = -3\sqrt{2/5}$, $x_2 = -\sqrt{2/5}$, and $x_3 = 5 - x_1 - x_2 = 5 + 4\sqrt{2/5}$.

Let us formalize the strategy we have used to find x and λ satisfying the conditions of Theorem 2.1 for a more general problem. To

$$\text{minimize } f(x) \text{ subject to } h(x) \leqslant b,\ x \in X \tag{2.2}$$

we proceed as follows:

1. Introduce a vector z of slack variables to obtain the equivalent problem
$$\text{minimize } f(x) \text{ subject to } h(x) + z = b,\ x \in X,\ z \geqslant 0.$$
2. Compute the Lagrangian $L(x, z, \lambda) = f(x) - \lambda^T(h(x) + z - b)$.
3. Define the set
$$Y = \{\lambda \in \mathbb{R}^m : \inf_{x \in X, z \geqslant 0} L(x, z, \lambda) > -\infty\}.$$
4. For each $\lambda \in Y$, minimize $L(x, z, \lambda)$ subject only to the regional constraints, i.e., find $x^*(\lambda), z^*(\lambda)$ satisfying
$$L(x^*(\lambda), z^*(\lambda), \lambda) = \inf_{x \in X, z \geqslant 0} L(x, z, \lambda). \tag{2.3}$$
5. Find $\lambda^* \in Y$ such that $(x^*(\lambda^*), z^*(\lambda^*))$ is feasible, i.e., such that $x^*(\lambda^*) \in X$, $z^*(\lambda^*) \geqslant 0$, and $h(x^*(\lambda^*)) + z^*(\lambda^*) = b$. By Theorem 2.1, $x^*(\lambda^*)$ is optimal for (2.2).

2.3 Complementary Slackness

It is worth pointing out a property known as *complementary slackness*, which follows directly from (2.3): for every $\lambda \in Y$ and $i = 1, \ldots, m$,

$$\begin{aligned} (z^*(\lambda))_i \neq 0 \quad &\text{implies} \quad \lambda_i = 0 \quad \text{and} \\ \lambda_i \neq 0 \quad &\text{implies} \quad (z^*(\lambda))_i = 0. \end{aligned}$$

Indeed, if the conditions were violated for some i, then the value of the Lagrangian could be reduced by reducing $(z^*(\lambda))_i$, while maintaining that $(z^*(\lambda))_i \geqslant 0$. This would contradict (2.3). Further note that $\lambda \in Y$ requires for each $i = 1, \ldots, m$ either that $\lambda_i \leqslant 0$ or that $\lambda_i \geqslant 0$, depending on the sign of b_i. In the case where where $\lambda_i \leqslant 0$, we for example get that

$$\begin{aligned} (h(x^*(\lambda^*)))_i < b_i \quad &\text{implies} \quad \lambda_i^* = 0 \quad \text{and} \\ \lambda_i^* < 0 \quad &\text{implies} \quad (h(x^*(\lambda^*)))_i = b_i. \end{aligned}$$

Slack in the corresponding inequalities $(h(x^*(\lambda^*)))_i \leqslant b_i$ and $\lambda_i^* \leqslant 0$ has to be complementary, in the sense that it cannot occur simultaneously in both of them.

Example 2.3. Consider the problem to

$$\begin{aligned} \text{minimize} \quad & x_1 - 3x_2 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leqslant 4 \\ & x_1 + x_2 \leqslant 2. \end{aligned}$$

By adding slack variables, we obtain the following equivalent problem:

$$\begin{aligned} \text{minimize} \quad & x_1 - 3x_2 \\ \text{subject to} \quad & x_1^2 + x_2^2 + z_1 = 4 \\ & x_1 + x_2 + z_2 = 2 \\ & z_1 \geqslant 0, z_2 \geqslant 0. \end{aligned}$$

The Lagrangian of this problem is

$$\begin{aligned} L(x, z, \lambda) &= x_1 - 3x_2 - \lambda_1(x_1^2 + x_2^2 + z_1 - 4) - \lambda_2(x_1 + x_2 + z_2 - 2) \\ &= \Big((1 - \lambda_2)x_1 - \lambda_1 x_1^2\Big) + \Big((-3 - \lambda_2)x_2 - \lambda_1 x_2^2\Big) - \lambda_1 z_1 - \lambda_2 z_2 + 4\lambda_1 + 2\lambda_2. \end{aligned}$$

Since $z_1 \geqslant 0$ and $z_2 \geqslant 0$, the terms $-\lambda_1 z_1$ and $-\lambda_2 z_2$ have a finite minimum only if $\lambda_1 \leqslant 0$ and $\lambda_2 \leqslant 0$. In addition, the complementary slackness conditions $\lambda_1 z_1 = 0$ and $\lambda_2 z_2 = 0$ must hold at the optimum.

Minimizing $L(x, z, \lambda)$ in x_1 and x_2 yields

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 1 - \lambda_2 - 2\lambda_1 x_1 = 0 \quad \text{and} \\ \frac{\partial L}{\partial x_2} &= -3 - \lambda_2 - 2\lambda_1 x_2 = 0, \end{aligned}$$

and we indeed obtain a minimum, because

$$\mathcal{H}L = \begin{pmatrix} \frac{\partial^2 L}{\partial x_1 \partial x_1} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2 \partial x_2} \end{pmatrix} = \begin{pmatrix} -2\lambda_1 & 0 \\ 0 & -2\lambda_1 \end{pmatrix}$$

is positive semidefinite when $\lambda_1 \leqslant 0$.

Setting $\lambda_1 = 0$ leads to inconsistent values for λ_2, so we must have $\lambda_1 < 0$, and, by complementary slackness, $z_1 = 0$. Also by complementary slackness, there are now two more cases to consider: the one where $\lambda_2 < 0$ and $z_2 = 0$, and the one where $\lambda_2 = 0$. The former case leads to a contradiction, the latter to the unique minimum at $x_1 = -\sqrt{2/5}$ and $x_2 = 3\sqrt{2/5}$.