11 The Transportation Algorithm

The particular structure of basic feasible solutions in the case of the transportation problem gives rise to a special interpretation of the simplex method. This special form is sometimes called the transportation algorithm.

11.1 Optimality Conditions

The Lagrangian of the transportation problem can be written as

$$\begin{aligned} L(x,\lambda,\mu) &= \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} + \sum_{i=1}^{n}\lambda_i\left(s_i - \sum_{j=1}^{m} x_{ij}\right) - \sum_{j=1}^{m}\mu_j\left(d_j - \sum_{i=1}^{n} x_{ij}\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}(c_{ij} - \lambda_i + \mu_j)x_{ij} + \sum_{i=1}^{n}\lambda_i s_i - \sum_{j=1}^{m}\mu_j d_j, \end{aligned}$$

where $\lambda \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ are Lagrange multipliers for the suppliers and consumers, respectively. Subject to $x_{ij} \geqslant 0$, the Lagrangian has a finite minimum if and only if

$$c_{ij} - \lambda_i + \mu_j \geqslant 0 \qquad \text{for } i = 1,\dots,n \text{ and } j = 1,\dots,m,$$

and at the optimum,

$$(c_{ij} - \lambda_i + \mu_j)x_{ij} = 0 \qquad \text{for } i = 1,\dots,n \text{ and } j = 1,\dots,m.$$

Together with feasibility of x, these dual feasibility and complementary slackness conditions are necessary and sufficient for optimality of x.

Note that the sign of the Lagrange multipliers can be chosen arbitrarily, and that this choice determines the form of the optimality conditions. The above choice is consistent with viewing demands as negative supplies.

11.2 The Simplex Method for the Transportation Problem

In solving instances of the transportation problem with the simplex method, a tableau of the following form will be useful:

$$\begin{array}{c|c|c|c|c} & \mu_1 & \cdots & \mu_m & \\ \hline \lambda_1 & x_{11}\ \boxed{c_{11}} & \cdots\ \boxed{\cdots} & x_{1m}\ \boxed{c_{1m}} & s_1 \\ \hline \vdots & \vdots\ \boxed{\vdots} & \ddots\ \boxed{\ddots} & \vdots\ \boxed{\vdots} & \vdots \\ \hline \lambda_n & x_{n1}\ \boxed{c_{n1}} & \cdots\ \boxed{\cdots} & x_{nm}\ \boxed{c_{nm}} & s_n \\ \hline & d_1 & \cdots & d_m & \end{array}$$

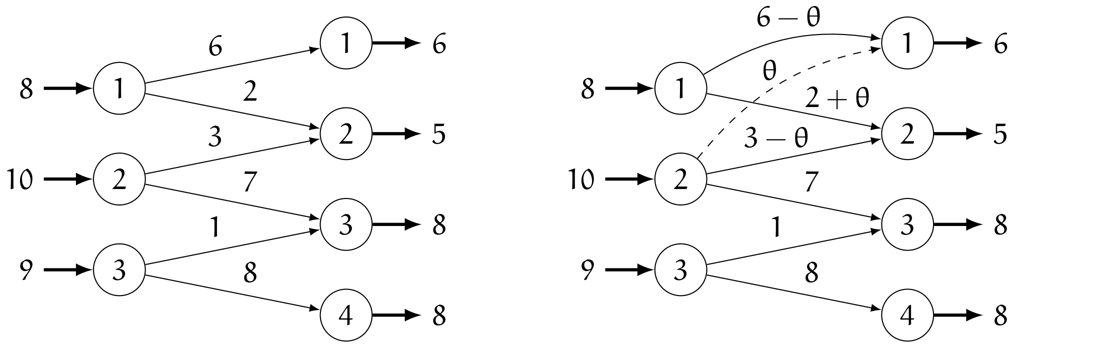

Figure 11.1: Initial basic feasible solution of an instance of the transportation problem (left) and a cycle along which the overall cost can be decreased (right)

Consider for example the Hitchcock transportation problem with three suppliers and four consumers given by the following tableau:

5	3	4	6	8
2	7	4	1	10
5	6	2	4	9
6	5	8	8	

Finding an initial BFS

An initial BFS can be found by iteratively considering pairs (i, j) of supplier i and consumer j, increasing x_{ij} until either the supply s_i or the demand d_j is satisfied, and moving to the next supplier in the former case or to the next consumer in the latter. Since $\sum_i s_i = \sum_j d_j$, this process is guaranteed to find a feasible solution. If at some intermediate step both supply and demand are satisfied at the same time, the resulting solution is degenerate. In general, degeneracies occur when a subset of the consumers can be satisfied exactly by a subset of the suppliers. In the example, we can start by setting $x_{11} = \min\{s_1, d_1\} = 6$, moving to consumer 2 and setting $x_{12} = 2$, moving to supplier 2 and setting $x_{22} = 3$, and so forth. The resulting flows are shown on the left of Figure 11.1.

Note that the initial BFS can be associated with a spanning tree (V, T) of the flow network where T is the set of edges visited by the above procedure. It then holds that $x_{ij} = 0$ when $(i, j) \notin T$, and complementary slackness dictates that $\lambda_i - \mu_j = c_{ij}$ when $(i, j) \in T$. By setting $\lambda_1 = 0$, we obtain a system of $n + m - 1$ linear equalities with $n + m - 1$ variables: each equality corresponds to an edge in T, each variable to a vertex in $(S \setminus \{1\}) \uplus C$. This system of equalities has a unique solution, allowing us to compute the values of the dual variables. We will see momentarily that every BFS can be associated with a spanning tree in this way. To verify dual feasibility, it will finally be convenient to write down $\lambda_i - \mu_j$ for $(i, j) \notin T$, and we do so in the upper right corner of the respective cells. For our example, we obtain the following tableau:

	-5	-3	0	-2	
0	$6\ \boxed{5}$	$2\ \boxed{3}$	$^{0}\ \boxed{4}$	$^{2}\ \boxed{6}$	8
4	$^{9}\ \boxed{2}$	$3\ \boxed{7}$	$7\ \boxed{4}$	$^{6}\ \boxed{1}$	10
2	$^{7}\ \boxed{5}$	$^{5}\ \boxed{6}$	$1\ \boxed{2}$	$8\ \boxed{4}$	9
	6	5	8	8	

Pivoting

If $c_{ij} \geqslant \lambda_i - \mu_j$ for all $(i, j) \notin T$, the current flow is optimal. Assume on the other hand that dual feasibility is violated for some edge $(i, j) \notin T$, and observe that this edge and the edges in T together form a unique cycle. In the absence of degeneracies the regional constraints for edges in T are not tight, so we can push flow around this cycle in order to increase x_{ij} and decrease the value of the Lagrangian. Due to the special structure of the network, this will alternately increase and decrease the flow for edges along the cycle until $x_{i'j'}$ becomes zero for some $(i', j') \in T$. We thus obtain a new BFS, and a new spanning tree in which (i', j') has been replaced by (i, j).

In our example dual feasibility is violated, for example, for $i = 2$ and $j = 1$. Edge $(2, 1)$ forms a unique cycle with the spanning tree T, and we would like to increase x_{21} by pushing flow along this cycle. In particular, increasing x_{21} by θ will increase x_{12} and decrease x_{11} and x_{22} by the same amount. The situation is shown on the right of Figure 11.1. If we increase x_{21} by the maximum amount of $\theta = 3$ and re-compute the values of the dual variables λ and μ, we obtain the following tableau:

	-5	-3	-7	-9
0	$3\ \boxed{5}$	$5\ \boxed{3}$	$^{7}\ \boxed{4}$	$^{9}\ \boxed{6}$
-3	$3\ \boxed{2}$	$^{0}\ \boxed{7}$	$7\ \boxed{4}$	$^{6}\ \boxed{1}$
-5	$^{0}\ \boxed{5}$	$^{-2}\ \boxed{6}$	$1\ \boxed{2}$	$8\ \boxed{4}$

Now, $c_{24} < \lambda_2 - \mu_4$, and we can increase x_{24} by 7 to obtain the following tableau, which satisfies $c_{ij} \geqslant \lambda_i - \mu_j$ for all $(i, j) \notin T$ and therefore yields an optimal solution:

	-5	-3	-2	-4
0	$3\ \boxed{5}$	$5\ \boxed{3}$	$^{2}\ \boxed{4}$	$^{4}\ \boxed{6}$
-3	$3\ \boxed{2}$	$^{0}\ \boxed{7}$	$^{-1}\ \boxed{4}$	$7\ \boxed{1}$
0	$^{5}\ \boxed{5}$	$^{3}\ \boxed{6}$	$8\ \boxed{2}$	$1\ \boxed{4}$

Let us summarize what we have done:

1. Find an initial BFS, and let T be the edges of the corresponding spanning tree.
2. Choose λ and μ such that $\lambda_1 = 0$ and $c_{ij} - \lambda_i + \mu_j = 0$ for all $(i, j) \in T$.
3. If $c_{ij} - \lambda_i + \mu_j \geqslant 0$ for all $(i, j) \in E$, the solution is optimal; stop.
4. Otherwise pick $(i, j) \in E$ such that $c_{ij} - \lambda_i + \mu_j < 0$, and push flow along the unique cycle in $(V, T \cup \{(i, j)\})$ until $x_{i'j'} = 0$ for some edge (i', j') in the cycle. Set T to $(T \setminus \{(i', j')\}) \cup \{(i, j)\}$ and go to Step 2.