17 Strategic Equilibrium

A pair of strategies $(x, y) \in X \times Y$ such that x is a best response to y and y is a best response to x is called an *equilibrium*. It is easily verified that both (C, D) and (D, C) are equilibria of the game of chicken, and we will see later that there is one more equilibrium, in which both players randomize between their two actions.

17.1 Equilibria of Matrix Games

The minimax theorem implies that every matrix game has an equilibrium, and in fact characterizes the set of equilibria of these games.

THEOREM 17.1. *A pair of strategies* $(x, y) \in X \times Y$ *is an equilibrium of the matrix game with payoff matrix* P *if and only if*

$$\begin{aligned} \min_{y' \in Y} p(x, y') &= \max_{x' \in X} \min_{y' \in Y} p(x', y') \quad \textit{and} \\ \max_{x' \in X} p(x', y) &= \min_{y' \in Y} \max_{x' \in X} p(x', y'). \end{aligned} \tag{17.1}$$

Proof. For all $(x, y) \in X \times Y$,

$$\min_{y' \in Y} \max_{x' \in X} p(x', y') \leqslant \max_{x' \in X} p(x', y) \geqslant p(x, y) \geqslant \min_{y' \in Y} p(x, y') \leqslant \max_{x' \in X} \min_{y' \in Y} p(x', y'),$$

and the first and last term are equal by Theorem 16.1.

If (x, y) is an equilibrium, the second and third inequality hold with equality. This means that the first and last inequality have to hold with equality as well, and (17.1) follows.

On the other hand, if (17.1) is satisfied, then the first and last inequality hold with equality. This means that the second and third inequality have to hold with equality as well, so (x, y) is an equilibrium. □

Other properties specific to matrix games are that all equilibria yield the same payoffs and that any pair of strategies of the two players, such that each of them is played in some equilibrium, is itself an equilibrium.

THEOREM 17.2. *Let* $(x, y), (x', y') \in X \times Y$ *be equilibria of the matrix game with payoff matrix* P. *Then* $p(x, y) = p(x', y')$, *and* (x, y') *and* (x', y) *are equilibria as well.*

Proof. Since equilibrium strategies are best responses to each other, we have that

$$p(x, y) \leqslant p(x, y') \leqslant p(x', y') \leqslant p(x', y) \leqslant p(x, y).$$

Since the first and last term are the same, the inequalities have to hold with equality and the first claim follows. Then,

$$
\begin{aligned}
p(x, y') &= p(x', y') \geqslant p(z, y') && \text{for all } z \in X, \\
p(x, y') &= p(x, y) \leqslant p(x, z) && \text{for all } z \in Y, \\
p(x', y) &= p(x, y) \geqslant p(z, y) && \text{for all } z \in X, \text{ and} \\
p(x', y) &= p(x', y') \geqslant p(x', z) && \text{for all } z \in X,
\end{aligned}
$$

where the inequalities hold because (x, y) and (x', y') are equilibria. Thus (x, y') and (x', y) are pairs of strategies that are best responses to each other, and the second claim follows as well. □

Theorems 16.1, 17.1, and 17.2 together also imply that the set of equilibria of a matrix game is convex.

17.2 Nash's Theorem

Many of the results concerning equilibria of matrix games do *not* carry over to bimatrix games, with the exception of existence.

THEOREM 17.3 (Nash, 1951). *Every bimatrix game has an equilibrium.*

We use the following result.

THEOREM 17.4 (Brouwer Fixed Point Theorem). *Let* $f : S \to S$ *be a continuous function, where* $S \subseteq \mathbb{R}^n$ *is closed, bounded, and convex. Then* f *has a fixed point.*

Proof of Theorem 17.3. Define X and Y as before, and observe that $X \times Y$ is closed, bounded, and convex. For $x \in X$ and $y \in Y$ define $s_i(x, y)$ and $t_j(x, y)$ as the additional payoff the two players could obtain by playing their ith or jth pure strategy instead of x or y, i.e.,

$$
\begin{aligned}
s_i(x, y) &= \max\{0, p(e_i^m, y) - p(x, y)\} && \text{for } i = 1, \dots, m \text{ and} \\
t_j(x, y) &= \max\{0, q(x, e_j^n) - q(x, y)\} && \text{for } j = 1, \dots, n,
\end{aligned}
$$

where e_ℓ^k denotes the ℓth unit vector in $\mathbb{R}^k$. Further define $f : (X \times Y) \to (X \times Y)$ by letting $f(x, y) = (x', y')$ with

$$
x_i' = \frac{x_i + s_i(x, y)}{1 + \sum_{k=1}^{m} s_k(x, y)} \qquad \text{and} \qquad y_j' = \frac{y_j + t_j(x, y)}{1 + \sum_{k=1}^{n} t_k(x, y)}
$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. Function f is continuous, so by Theorem 17.4 is must have a fixed point, i.e., a pair of strategies $(x, y) \in X \times Y$ such that $f(x, y) = (x, y)$.

Further observe that there has to exist $i \in \{1, \dots, m\}$ such that $x_i > 0$ and $s_i(x, y) = 0$, since otherwise

$$p(x, y) = \sum_{k=1}^{m} x_k p(e_k^m, y) > \sum_{k=1}^{m} x_k p(x, y) = p(x, y).$$

Therefore, and since (x, y) is a fixed point,

$$x_i = \frac{x_i + s_i(x, y)}{1 + \sum_{k=1}^{m} s_k(x, y)}$$

and thus

$$\sum_{k=1}^{m} s_k(x, y) = 0.$$

This means that for $k = 1, \dots, m$, $s_k(x, y) = 0$, and therefore

$$p(x, y) \geqslant p(e_k^m, y).$$

It follows that

$$p(x, y) \geqslant p(x', y) \quad \text{for all } x' \in X.$$

An analogous argument shows that $q(x, y) \geqslant q(x, y')$ for all $y' \in Y$, so (x, y) must be an equilibrium. □

Our requirement that a bimatrix game has a finite number of actions is crucial for this result. This can be seen very easily by considering a game where the set of actions of each player is the set of natural numbers, and players get a payoff of 1 if they choose a number that is greater than the one chosen by the other player, and zero otherwise.

17.3 The Complexity of Finding an Equilibrium

The proof of Theorem 17.3 relies on fixed points of a continuous function and does not give rise to a finite method for finding an equilibrium. Quite surprisingly, equilibrium computation turns out to be more or less a combinatorial problem.

Define the *support* of strategy $x \in X$ as $S(x) = \{i \in \{1, \dots, m\} : x_i > 0\}$, and that of strategy $y \in Y$ as $S(y) = \{j \in \{1, \dots, m\} : y_j > 0\}$. It is easy to see that a mixed strategy is a best response if and only if all pure strategies in its support are best responses: if one of them was not a best response, then the payoff could be increased by reducing the probability of that strategy, and increasing the probabilities of the other strategies in the support appropriately. In other words, randomization over the support of an equilibrium does not happen for the player's own sake, but to allow the other player to respond in a way that sustains the equilibrium.

It also follows from these considerations that finding an equilibrium boils down to finding its supports. Once the supports are known, the precise strategies can be computed by solving a set of equations, which in the two-player case are linear. For supports of sizes k and ℓ, there is one equation for each player stating that the probabilities sum up to one, and $k-1$ or $\ell-1$ equations, respectively, stating that the expected payoff is the same for every pure strategy in the support. Solving these $k+\ell$ equations in $k+\ell$ variables yields k values for player 1 and ℓ values for player 2. If the solution corresponds to a strategy profile and expected payoffs are maximized by the pure strategies in the support, then an equilibrium has been found. An equilibrium with supports of size two in the game of chicken would have to satisfy $x_1 + x_2 = 1$, $y_1 + y_2 = 1$, $2x_1 + 1x_2 = 3x_1 + 0x_2$, and $2y_1 + 1y_2 = 3y_1 + 0y_2$. The unique solution, $x_1 = x_2 = y_1 = y_2 = 1/2$, also satisfies the additional requirements and therefore is an equilibrium. No equilibrium with full supports exists in the prisoner's dilemma, because the corresponding system of equalities does not have a solution.

A basic procedure for finding an equilibrium, and in fact all equilibria, is to iterate over all possible supports and check for each of them whether there is an equilibrium with that support. The running time of this method is finite, but clearly exponential in general. It is natural to ask whether there is a hardness result that stands in the way of a polynomial-time algorithm. While the equilibrium condition can easily be verified for a given pair of strategies, which implies membership in NP, the notion of NP-hardness seems inappropriate: equilibria always exist and the decision problem is therefore trivial. On the other hand, NP-hardness follows immediately if the problem is modified slightly to obtain a non-trivial decision problem.

THEOREM 17.5. *Given a bimatrix game, it is NP-complete to decide whether it has at least two equilibria; an equilibrium in which the expected payoff of the row player is at least a given amount; an equilibrium in which the expected sum of payoff of the two players is a least a given amount; an equilibrium with supports of a given minimum size; an equilibrium whose support includes a given pure strategy; or an equilibrium whose support does not include a given pure strategy.*

In the next lecture we will consider an algorithm that searches the possible supports in a more organized way. This algorithm also provides an alternative, combinatorial, proof of existence, and will lead us to the appropriate complexity class for the problem of finding an equilibrium.