Mathematics of Operational Research **Example Sheet 3**
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1 Let $(x, y), (x', y') \in X \times Y$ be equilibria of the matrix game with payoff matrix P. Show that $p(x, y) = p(x', y')$, and that (x, y') and (x', y) are equilibria as well.

2 Consider the following bimatrix game:

$(2,6)$	$(4,2)$
$(6,0)$	$(0,4)$

(a) Compute all equilibria of this game, as well as the maximin strategies for both players.

(b) Discuss your findings, and show that they can be attributed to the failure of a particular theorem in the lecture notes.

3 Use the Lemke-Howson algorithm to show that both players receive an expected payoff of $2/3$ in the unique equilibrium of the following bimatrix game:

$(0,1)$	$(1,0)$	$(0,-1)$
$(2,0)$	$(0,2)$	$(-1,1)$

4 Use LP duality to prove the Bondareva-Shapley Theorem.

5 Prove that the nucleolus of any coalitional game is a singleton. To this end, consider two imputations $x \neq y$ in the nucleolus and show that $E((x+y)/2)$ is lexicographically smaller than $E(x)$.

6 A coalitional game is

- *convex* if $v(S \cup T) \geqslant v(S) + v(T) - v(S \cap T)$ for all $S, T \subseteq N$,
- *superadditive* if $v(S \cup T) \geqslant v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$, and
- *additive* if $v(S \cup T) = v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$.

(a) Prove or disprove that every additive game is convex.

(b) Prove or disprove that every convex game is superadditive.

(c) Prove or disprove that every superadditive game has a nonempty core.

7 For an undirected graph (V, E) and a weight function $w : E \to \mathbb{R}$, consider the coalitional game with set V of players and characteristic function v given by

$$v(S) = \sum_{\{i,j\} \subseteq S} w(\{i, j\}).$$

Show that the Shapley value of player $i \in V$ in this game is

$$\frac{1}{2} \sum_{j \in V \setminus \{i\}} w(\{i, j\}).$$

8 Consider a jury system with twelve jurors in which a defendant is found guilty if voted guilty by ten or more of the jurors.

(a) Represent this jury system as a coalitional game where $v(S) = 1$ if the defendant is found guilty if voted guilty by all members of S, and $v(S) = 0$ otherwise. Show that the core of this game is empty, and that the nucleolus and the vector of Shapley values are both $(1/12, 1/12, \dots, 1/12)$.

Now assume that there is a judge in addition to the jurors, and that to be found guilty the defendant in particular has to be voted guilty by the judge.

(b) Represent the new situation as a coalition game, and determine the core, the nucleolus, and the Shapley value of each player.

9 Consider the following bimatrix game:

$(4,1)$	$(1,0)$
$(-1,2)$	$(2,3)$

(a) Determine the Nash bargaining solution of this game.

(b) Show that the game can be interpreted as a two-player coalitional game with characteristic function v given by $v(\{1\}) = 3/2$, $v(\{2\}) = 1$, and $v(\{1,2\}) = 5$. Determine the core, the nucleolus, and the Shapley value of each player.

10 Call a matching procedure strategyproof if it is a weakly dominant strategy for every agent to submit its true preferences. Show that a matching procedure that always produces a stable matching cannot be strategyproof.

11 Show that Kemeny's rule fails independence of irrelevant alternatives.

12 In *Borda's rule*, an alternative receives $m - 1$ points for each voter who ranks it first, $m - 2$ points for each voter who ranks it second, and so forth. A *Borda winner* or *loser* then is an alternative with a maximum or minimum overall number of points.

(a) Show that Borda's rule is not Condorcet consistent.

(b) Show that a Condorcet winner is never a Borda loser, and a Condorcet loser is never a Borda winner. To this end, show that the score of an arbitrary alternative a is equal to $\sum_{b \in A \setminus \{a\}} \#(a, b)$, where $\#(a, b) = |\{i \in N : a \succ_i b\}|$.

(c) Combine this insight with an SCF defined in the lecture notes to obtain an SCF that is Condorcet consistent.

13 *Plurality with runoff* chooses an alternative that is preferred by a majority of the voters among two alternatives ranked first by the largest number of voters. *Copeland's rule* chooses an alternative that wins a maximum number of pairwise comparisons, i.e., an alternative a for which $C(a) = \max_{a' \in A} C(a')$, where $C(a') = |\{b \in A : \#(a', b) > n/2\}| - |\{b \in A : \#(b, a') > n/2\}|$. *Maximin* chooses an alternative for which the largest pairwise defeat is as small as possible, i.e., an alternative $a \in A$ such that $\min_{b \in A} \#(a, b) = \max_{a' \in A} \min_{b \in A} \#(a', b)$, where $\#(a, b) = |\{i \in N : a \succ_i b\}|$.

Determine the alternatives selected by plurality, plurality with runoff, Borda's rule, Copeland's rule, Kemeny's rule, maximin, and STV in the following situation with 100 voters and 5 alternatives a, b, c, d, and e:

33	16	3	8	18	22
a	b	c	c	d	e
b	d	d	e	e	c
c	c	b	b	c	b
d	e	a	d	b	d
e	a	e	a	a	a

14 Consider a situation with n agents and k identical copies of an item, and assume that agent i has value $v_i \geqslant 0$ if it receives at least one copy, and value 0 otherwise. Derive the VCG mechanism with Clarke pivot rule for this problem. Give a succinct description of the mechanism and explain from first principles why it is strategyproof.

15 Assume that the residents of an apartment complex have to decide whether to build a shared swimming pool. Resident i has value $v_i \geqslant 0$ if the pool is built, and value 0 otherwise. The cost of the pool is C, so it should be built if $\sum_i v_i \geqslant C$.

(a) Derive the VCG mechanism with Clarke pivot rule for this problem. Show that this mechanism does not collect enough money to pay for the pool unless $\sum_i v_i = C$, or unless there is a single resident i such that $v_j \geqslant C$ if $j = i$ and $v_j = 0$ otherwise. Also show that the mechanism is susceptible to collusion, i.e., that a coalition of residents may change their revealed values such that the utility of every member of the coalition increases.

Now assume that in addition to the decision whether the pool is built, we can also decide which residents are allowed to use it. Consider a mechanism that works as follows. Initially, all residents submit their value for the pool. Then the mechanism proceeds in rounds and successively excludes residents from using the pool if it was built. Assume that in a certain round, the set of remaining residents is S. If for all $i \in S$, $v_i \geqslant C/|S|$, then the pool is built, and each resident in S pays $C/|S|$ and is allowed to use the pool. Otherwise all residents i for which $v_i < C/|S|$ are excluded, and the mechanism continues. If at some point all residents have been excluded, the pool is not built.

(b) Show that this mechanism is strategyproof.

(c) Show that the outcome function of this mechanism is not an affine maximizer. Why is this not a contradiction to Robert's Theorem?