Mathematics of Operational Research **Example Sheet 1**
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1 Show that the optimization problem to

$$\begin{aligned} \text{maximize} \quad & -2x_1^2 - x_2^2 + x_1x_2 + 8x_1 + 3x_2 \\ \text{subject to} \quad & 3x_1 + x_2 = 10 \end{aligned}$$

has an optimal solution at $(x_1, x_2) = (69/28, 73/28)$.

2 Suppose that f and h are continuously differentiable on $\mathbb{R}^n$, and that there exist unique functions $x^* : \mathbb{R}^m \to \mathbb{R}^n$ and $\lambda^* : \mathbb{R}^m \to \mathbb{R}^m$ such that for each $b \in \mathbb{R}^m$, $h(x^*(b)) = b$, $\lambda^*(b) \leqslant 0$ and $f(x^*(b)) = \phi(b) = \inf\{f(x) - \lambda^*(b)^T(h(x) - b) : x \in \mathbb{R}^n\}$. Show that if x^* and λ^* are continuously differentiable, then

$$\frac{\partial \phi}{\partial b_i}(b) = \lambda_i^*(b).$$

To this end, show that

$$\begin{aligned} \frac{\partial \phi(b)}{\partial b_i} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}(x^*(b)) - \lambda^*(b)^T \frac{\partial h}{\partial x_j}(x^*(b)) \right) \frac{\partial x_j^*}{\partial b_i}(b) \\ - (h(x^*(b)) - b) \frac{\partial \lambda^*(b)^T}{\partial b_i} + \lambda^*(b)^T \frac{\partial b}{\partial b_i}, \end{aligned}$$

and argue that the first two terms on the right-hand side are zero.

3 Find an optimal solution of the problem to

$$\begin{aligned} \text{maximize} \quad & 2\tan^{-1} x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leqslant b_1 \\ & -\log x_2 \leqslant b_2 \\ & x_1, x_2 \geqslant 0, \end{aligned}$$

where b_1 and b_2 are constants such that $b_1 - e^{-b_2} \geqslant 0$. You may want to distinguish the cases in which the Lagrange multiplier for the second constraint is equal to 0 and greater than 0, respectively.

4 Show that the dual of the dual of a linear program is equivalent to the primal.

5 Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, and consider the linear programs

$$\max\{\, 0^T x : Ax = b,\ x \geqslant 0 \,\} \quad \text{and} \tag{1}$$

$$\min\{\, y^T b : y^T A \geqslant 0^T \,\}. \tag{2}$$

(a) Show that (2) is the dual of (1).

(b) Show that (1) is feasible if and only if (2) is bounded.

(c) Prove Farkas' Lemma, which states that exactly one of the following is true:

1. There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geqslant 0$.
2. There exists $y \in \mathbb{R}^m$ such that $y^T A \geqslant 0$ and $y^T b < 0$.

6 Show that adding slack variables to a linear program does not change the extreme points of the feasible set, i.e., that $x^* \in \mathbb{R}^n$ is an extreme point of $\{x \in \mathbb{R}^n : x \geqslant 0, Ax \leqslant b\}$ if and only if for some $z^* \in \mathbb{R}^m$, $\binom{x^*}{z^*}$ is an extreme point of $\{\binom{x}{z} \in \mathbb{R}^{n+m} : \binom{x}{z} \geqslant 0, Ax + z = b\}$.

7 Show that a linear program that is feasible and bounded has an optimal solution that is a BFS. You may want to consider an optimal solution that is not basic and show that there must exist an optimal solution with strictly fewer non-zero entries.

8 Consider the problem to

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leqslant 4 \\ & x_1 + 2x_2 \leqslant 4 \\ & x_1 - x_2 \leqslant 1 \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

(a) Solve the problem graphically in the plane.

(b) Introduce slack variables x_3, x_4, and x_5 and write the problem in equality form. How many basic solutions are there? Determine the value of $x = (x_1, \dots, x_5)^T$ and of the objective function at each of the basic solutions. Which of the basic solutions are feasible? Are all basic solutions non-degenerate?

(c) Write down the dual problem in equality form using slack variables λ_4 and λ_5, and determine the value of $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ and of the objective function at each of the basic solutions of the dual. Which of these basic solutions are feasible?

(d) Write down the complementary slackness conditions for the problem, and show that for each basic solution of the primal there is exactly one basic solution of the dual such that the two have the same value and satisfy complementary slackness. How many of these pairs are feasible for both primal and dual?

(e) Solve the problem using the simplex method. Start from the basic feasible solution where $x_1 = x_2 = 0$, and try both choices for a variable to enter into the basis. How are the entries in the last row of the various tableaus related to the appropriate basic solutions of the dual?

9 Use the two-phase simplex method to show that the linear program

$$\begin{aligned} \text{minimize} \quad & 4x_1 + 4x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leqslant 2 \\ & 2x_1 + x_2 \leqslant 3 \\ & 2x_1 + x_2 + 3x_3 \geqslant 3 \\ & x_1, x_2, x_3 \geqslant 0 \end{aligned}$$

has an optimal solution at $x = (0, 0, 1)$.

10 Consider the integer program (IP)

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2 \\ \text{subject to} & -3x_1 + 4x_2 \leqslant 4 \\ & 3x_1 + 2x_2 \leqslant 11 \\ & 2x_1 - x_2 \leqslant 5 \\ & x_1, x_2 \geqslant 0,\ x_1, x_2 \in \mathbb{Z}. \end{array}$$

(a) Use the simplex method to solve the LP relaxation of the IP and verify that the final tableau looks as follows:

x_1	x_2	z_1	z_2	z_3	
0	1	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{5}{2}$
1	0	$-\frac{1}{9}$	$\frac{2}{9}$	0	2
0	0	$\frac{7}{18}$	$-\frac{5}{18}$	1	$\frac{7}{2}$
0	0	$-\frac{2}{9}$	$-\frac{5}{9}$	0	-7

(b) Explain why the optimal solution of the IP must satisfy $x_2 \leqslant 2$.

(c) Use the cutting plane method to solve the IP.

11 A Hamiltonian cycle of a graph is a cycle that visits every node. The directed Hamiltonian cycle problem asks whether a given directed graph has a Hamiltonian cycle.

(a) Show that this problem is in NP.

(b) Give a reduction from the Boolean satisfiability problem to show that the problem is also NP-hard. For each variable of a given Boolean formula, arrange an appropriate number of nodes from left to right, and connect them in such a way that there are exactly two paths that visit all of them, one from left to right and one from right to left, corresponding to setting the variable to true or false. Now represent each clause by one node, and connect this node to the chain of nodes of every variable contained in the clause, in such a way that the node can be visited while traversing the nodes for a particular variable if and only if the variable has been set in a way that satisfies the clause.

(c) Show that the traveling salesman problem is NP-hard, by observing that the undirected Hamiltonian cycle problem is a special case of it and reducing the directed Hamiltonian cycle problem to the undirected one. The key element of the reduction is to replace every node in a directed graph by three nodes in an undirected one, such that there is a direct correspondence between paths in the two graphs.