Optimal Impartial Selection and the Power of Up to Two Choices

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Optimal Impartial Selection

- Select members of a set of agents based on nominations by agents from the same set
- Applications
  - selection of representatives
  - award of prizes
  - assignment of responsibilities
  - peer review: papers, research proposals, ...
- Assumption: agents are impartial to the selection of *other* agents
  - will reveal their opinion truthfully...
  - as long as it does not affect their own chance of selection
- Goal: preserve impartiality, select agents with many nominations

Formal Model

- Set $\mathcal{G}_N$ of graphs $G = (N, E)$ without self-loops
  vertices represent agents, $(i, j) \in E$ means i nominates j
- $\delta^-_S(X, G) = \left|\left\{(i, j) \in E : G = (N, E), i \in S, j \in X\right\}\right|$
  number of nominations agents in X receive from agents in S
- $\Delta_k(G) = \max_{X \in \binom{N}{k}} \delta^-_N(X, G)$

- k-selection mechanism $f : \mathcal{G}_N \to [0, 1]^{\cup_{\ell \leq k} \binom{N}{\ell}}$ with $\sum_X (f(G))_X = 1$
- deterministic if $(f(G))_X \in \{0, 1\}$ for all G and X
- exact if $f(G)_X = 0$ when $|X| < k$
- impartial if $\sum_{X, i \in X} (f((N, E)))_X = \sum_{X, i \in X} (f((N, E')))_X$ when $E \ominus E' \subseteq \{i\} \times N$
- α-optimal if for $\alpha \leq 1$ and all G, $\mathbb{E}_{X \sim f(G)}[\delta^-_N(X, G)] \geq \alpha \cdot \Delta_k(G)$

State of the Art and Our Contribution

- Exact deterministic mechanisms are useless
  - cannot be α-optimal for any k and $\alpha > 0$ (Alon et al., 2011)
  - fail very basic axioms for $k = 1$ (Holzman & Moulin, 2013)
- Case where $k = 1$ is well understood
  - no mechanism better than $1/2$-optimal (Alon et al., 2011)
  - $1/2$-optimality is possible (Fischer & Klimm, 2015)
- Asymptotically optimal mechanisms
  - for selecting many agents (Alon et al., 2011)
  - when agents receive many nominations (Bousquet et al., 2014)
- We study exact and inexact k-selection mechanisms, starting from $k = 2$

The Permutation Mechanism (Fischer & Klimm)

$(\pi_1, \ldots, \pi_n) \leftarrow$ random permutation of N; let $\pi_{<j} = \{\pi_1, \pi_2, \ldots, j\}$

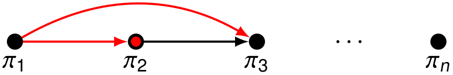
$i \leftarrow \pi_1$; $d \leftarrow 0$

for $j = \pi_2, \ldots, \pi_n$

  if $\delta^-_{\pi_{<j} \setminus \{i\}}(j, G) \geq d$

    $i \leftarrow j$, $d \leftarrow \delta^-_{\pi_{<j}}(j, G)$

return i



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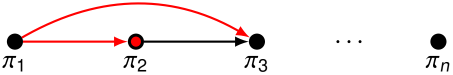

- Impartial
- 1/2-optimal

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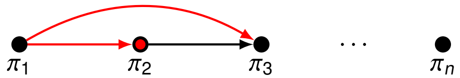

- Impartial: $\Xi_\pi(G)$ is impartial for any π
- 1/2-optimal: if $\Xi_\pi(G) = i$, then $\delta^-_{\pi_{<i}}(i, G) \geq \max_{j \in N} \delta^-_{\pi_{<j}}(j, G)$.

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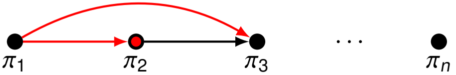

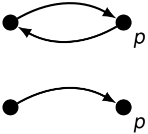

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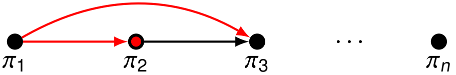

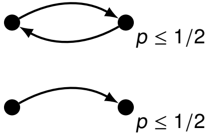

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Deterministic 2-Selection

$(\pi_1, \ldots, \pi_n) \leftarrow$ arbitrary permutation of N; let $\bar{\pi} = (\pi_n, \ldots, \pi_1)$

$i_1 \leftarrow \Xi_\pi(G)$

$i_2 \leftarrow \Xi_{\bar{\pi}}(G)$

return $\{i_1, i_2\}$

- Impartial: Ξ is impartial, union maintains impartiality
- 1/2-optimal: consider $i^* \in N$ with $\delta^-(i^*) = \Delta_1$

$$\begin{aligned}\delta^-(\{i_1, i_2\}) &\geq \delta^-_{\pi_{<i_1}}(i_1) + \delta^-_{\bar{\pi}_{<i_2}}(i_2) \\ &\geq \delta^-_{\pi_{<i^*}}(i^*) + \delta^-_{\bar{\pi}_{<i^*}}(i^*) = \delta^-(i^*) = \Delta_1 \geq \frac{1}{2}\Delta_2\end{aligned}$$

- Not exact: possibly $i_1 = i_2$
- No mechanism better than 1/2-optimal

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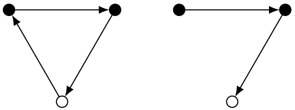

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$$\geq \delta^-_{\pi_{<i^*}}(i^*) + \delta^-_{\bar{\pi}_{<i^*}}(i^*) = \delta^-(i^*) = \Delta_1 \geq \frac{1}{2}\Delta_2$$

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Randomized 2-Selection

$(\pi_1, \dots, \pi_n) \leftarrow$ random permutation of N; let $\bar{\pi} = (\pi_n, \dots, \pi_1)$

$i_1 \leftarrow \Xi_\pi(G)$

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return $\{i_1, i_2\}$

- Impartial, not exact
- 2/3-optimal: consider $i_1^*, i_2^* \in N$ with $\delta^-(i_1^*) + \delta^-(i_2^*) = \Delta_2$

$$\begin{aligned}\mathbb{E}\Big[\delta^-(\{i_1, i_2\})\Big] &\geq \mathbb{E}\Big[\delta^-_{\pi_{<i_1}}(i_1) + \delta^-_{\bar{\pi}_{<i_2}}(i_2)\Big] \\ &\geq \mathbb{E}\Big[\max\Big\{\delta^-_{\pi_{<i_1^*}}(i_1^*), \delta^-_{\pi_{<i_2^*}}(i_2^*)\Big\} + \max\Big\{\delta^-_{\bar{\pi}_{<i_1^*}}(i_1^*), \delta^-_{\bar{\pi}_{<i_2^*}}(i_2^*)\Big\}\Big] \\ &\geq \cdots \geq \tfrac{2}{3}\delta^-(i_1^*) + \tfrac{2}{3}\delta^-(i_2^*)\end{aligned}$$

- No mechanism better than 3/4-optimal

Randomized 2-Selection

$(\pi_1, \dots, \pi_n) \leftarrow$ random permutation of N; let $\bar{\pi} = (\pi_n, \dots, \pi_1)$

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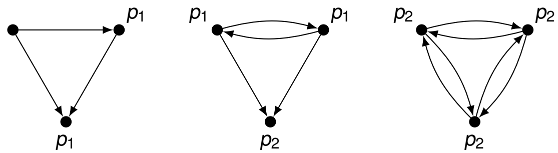

- Impartial, not exact
- 2/3-optimal: consider $i_1^*, i_2^* \in N$ with $\delta^-(i_1^*) + \delta^-(i_2^*) = \Delta_2$

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Exact 2-Selection

$(A_1, A_2) \leftarrow$ random partition of N with $A_1 \neq \emptyset$

$(\pi_1, \ldots, \pi_{|N|}) \leftarrow$ random permutation of N

$i_1 \leftarrow \Xi_{\pi, A_1}(G)$

$i_2 \leftarrow \Xi_{\pi, A_2}(G)$, or random element of $A_1 \setminus \{i_1\}$ if $A_2 = \emptyset$

return $\{i_1, i_2\}$

$$\left.\begin{array}{l} i \leftarrow \pi_1;\ d \leftarrow 0 \\ \text{for } j = 2, \ldots, k \\ \quad \text{if } \pi_j \in A \text{ and } \delta^-_{(N \setminus A) \cup (\pi_{<j} \setminus \{i\})}(\pi_j) \geq d \\ \qquad i \leftarrow \pi_j,\ d \leftarrow \delta^-_{(N \setminus A) \cup \pi_{<j}}(\pi_j) \\ \text{return } i \end{array}\right\} \Xi_{\pi, A}(G)$$

- Impartial, 7/12-optimal, no mechanism better than 2/3-optimal

Summary and Open Problems

2-selection

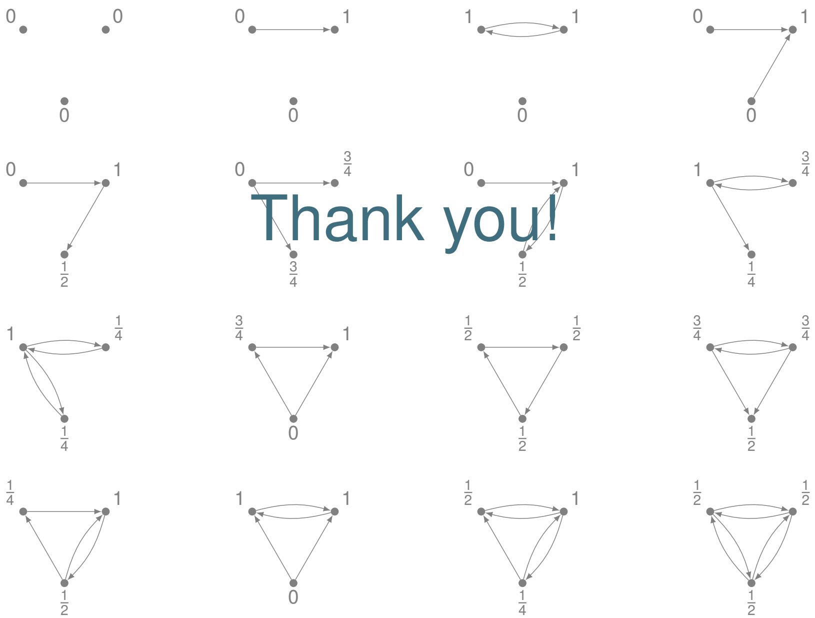
deterministic	randomized exact	randomized
$1/2$	$[7/12, 2/3]$	$[2/3, 3/4]$

- gaps may be due to universal impartiality

k-selection

- upper bounds of around $(k+1)/(k+2)$, probably tight
- no good lower bounds, especially for deterministic case
- no obvious analog for two directions of a permutation
- asymptotically optimal mechanisms exist

Thank you!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