Complexity Results for Some Classes of Strategic Games

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Outline

1. Problem: Games, Solutions, and Complexity
2. Results: An Overview
3. Example: Pure Nash Equilibria of Anonymous Games

Game Theory

- A mathematical theory of strategic interaction
- John von Neumann, Oskar Morgenstern: Theory of Games and Economic Behavior (1944)

- Non-cooperative game theory
  - Different outcomes depending on choices of several individuals (players)
  - Disagreement about quality of the outcomes

- Applications in economics, political science, biology, . . .
- In computer science: analysis of electronic markets, the Internet, social networks, . . .

Normal-form Games

- Normal-form game $\Gamma = (N, (A_i)_{i \in N}, (p_i)_{i \in N})$
  - N a (finite) set of *players*
  - A_i a (finite) set of *actions* for player i
  - $p_i : \bigtimes_{j \in N} A_j \to \mathbb{R}$ a *payoff function* for player i

	O	F
O	$(2, 1)$	$(0, 0)$
F	$(0, 0)$	$(1, 2)$

- Rational players: maximize their own payoff
- Strategy of player i: probability distribution $s_i \in S_i = \Delta(A_i)$
- Strategy profile: vector $s \in S = \bigtimes_{j \in N} S_j$ of strategies

From Games to Solutions

- Solution concepts single out “interesting” strategy profiles
- Nash equilibrium: profile of strategies that are mutual best responses
- Formally: $s \in S$ such that for all $i \in N$, $a \in A_i$, $p_i(s) \geq p_i(s_{-i}, a)$

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- Existence guaranteed (Nash, 1950);

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	H	T
H	$(1,-1)$	$(-1,1)$
T	$(-1,1)$	$(1,-1)$

- Existence guaranteed (Nash, 1950); not so for *pure* equilibrium

Towards Algorithmic Game Theory

- Nobel laureate Robert Aumann: “A solution concept must be calculable, otherwise you are not going to use it.”
- Still, computational complexity of finding solutions has received fairly little attention in traditional game theory
- Possible reason: the right tools were missing

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- Computational complexity theory
  - Classes of problems with similar resource requirements
  - P (efficiently solvable) vs. NP (efficiently verifiable)
  - NP-hard: not in P if P $\neq$ NP

The Complexity of Nash Equilibrium

- Pure Nash equilibrium: decision problem
  - decidable by enumeration of action profiles, complexity depends on *representation*
  - in P when games are given explicitly
  - potentially NP-hard given succinct description of games with many players
- Nash equilibrium: search problem, solution guaranteed to exist
  - PPAD-complete, even for $|N| = 2$ (Chen and Deng, 2005)
  - known algorithms (*e.g.*, Lemke's algorithm) have exponential worst-case running time

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- Maybe real-world games are not “general”
- Evidence: number of outcomes in general games may be exponential in $|N|$, even if $|A| = 2$
- Could not even be *played* efficiently

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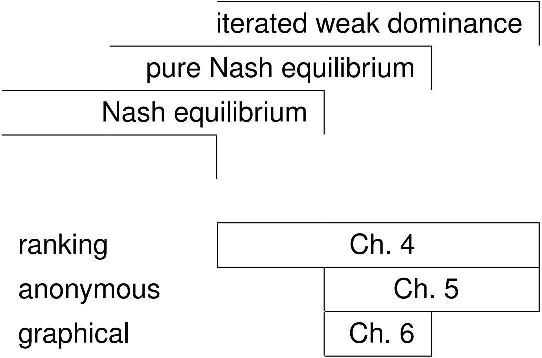
- Maybe real-world games are not "general"
- Evidence: number of outcomes in general games may be exponential in $|N|$, even if $|A| = 2$
- Could not even be *played* efficiently
- Consider restricted classes of (multi-player) games with properties found in the real world

Some Classes of Strategic Games

- Ranking Games
  - outcomes are rankings of the players
  - only performance *relative to the others* matters
  - examples: parlor games, economic scenarios
- Anonymous Games
  - other players are similar, cannot be distinguished
  - payoff only depends on *how many* of them play each action
  - example: large open systems (e.g., the Internet)
- Graphical Games
  - payoff depends only on players in a local neighborhood
  - examples: networks (computer or social)

Overview of Results



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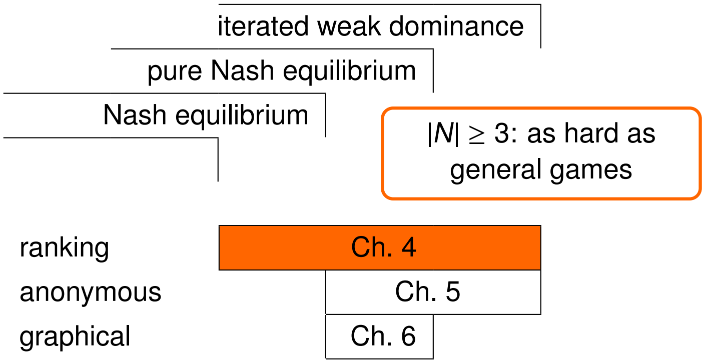

Brandt, Fischer, Harrenstein, Shoham: *Ranking Games*, Artificial Intelligence, 2009 (also 21st AAAI, 2006, and 20th IJCAI, 2007)

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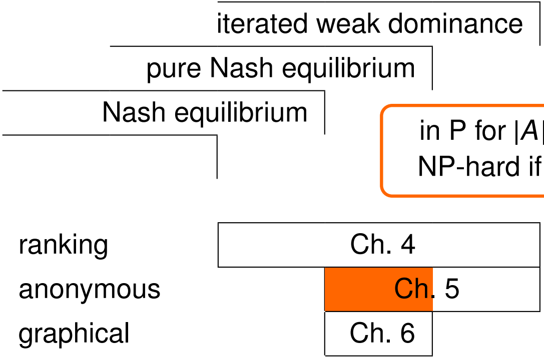

in P for $|A| = const.$
NP-hard if $|A|$ grows

Brandt, Fischer, Holzer: *Symmetries and the complexity of pure Nash equilibrium*, JCSS, 2009 (also 24th STACS, 2007)

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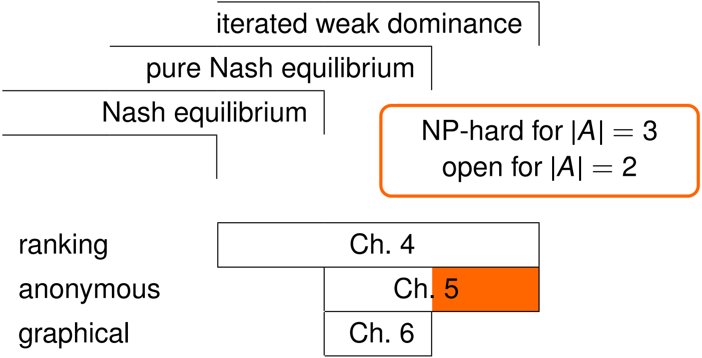

Brandt, Fischer, Holzer: *On iterated dominance, matrix elimination, and matched paths*, 2008

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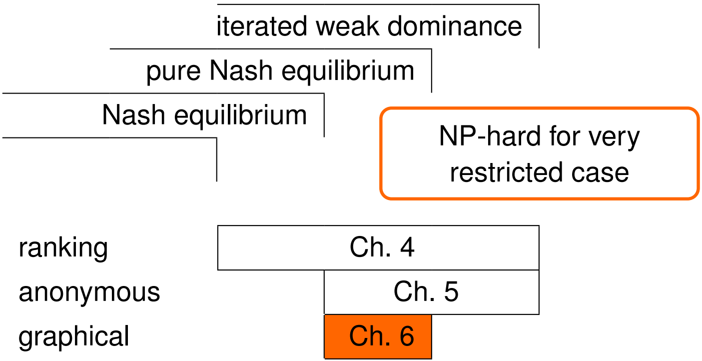

Fischer, Holzer, Katzenbeisser: *The influence of neighbourhood and choice on the complexity of finding pure Nash equilibria*, IPL, 2006

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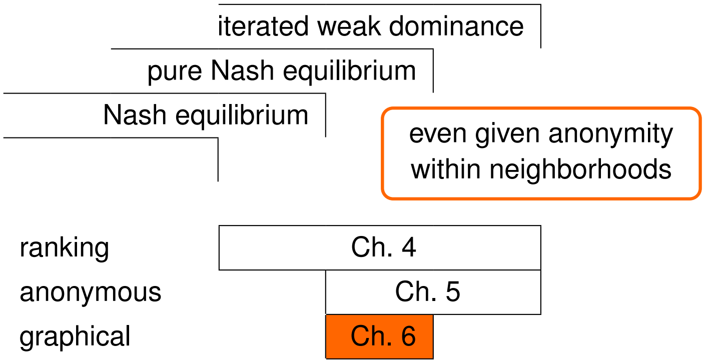

Brandt, Fischer, Holzer: *Equilibria of graphical games with symmetries*, 4th WINE, 2008

Two More Solution Concepts

- Nash equilibrium commonly criticized
- Indifference between actions played and not played
- Requires randomization (with irrational weights if $|N| \geq 3$)

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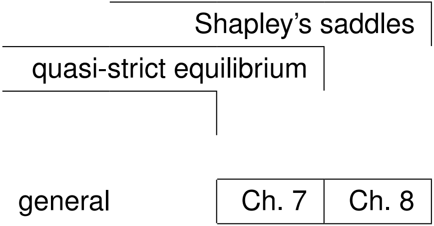

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Shapley's saddles

quasi-strict equilibrium

NP-hard for $|N| \geq 3$
(need not exist)

general | Ch. 7 | Ch. 8

Brandt, Fischer: *On the hardness and existence of quasi-strict equilibria*, 1st SAGT, 2008

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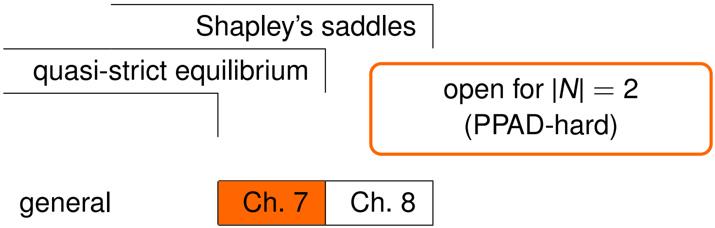

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quasi-strict equilibrium

strict: in P
weak: NP-hard for $|N| \geq 2$

general | Ch. 7 | Ch. 8

Brandt, Brill, Fischer, Harrenstein: *Computational aspects of Shapley's saddles*, 8th AAMAS, 2009

Anonymous Games

- No distinction between other players
- All players have same set A of actions
- Payoff determined by
  - own action
  - number of other players playing each action
- Useful concept: *commutative image* of action profile s

$$\#(s) = (|\{i \in N \mid s_i = a\}|)_{a \in A}$$

An Example

p_1:

	0	1	2
0	0	0	0
1	1	0	1

p_2:

	0	1	2
0	1	1	1
1	0	0	0

p_3:

	0	1	2
0	1	1	0
1	0	0	1

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- Equilibrium property not determined by commutative image

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Pure Nash Equilibria of Anonymous Games

Theorem: In anonymous games with a constant number of actions, existence of a pure Nash equilibrium can be decided in polynomial time.

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Proof: by reduction to perfect matchings of a bipartite graph

- Fix a commutative image x (only polynomially many)
- Left side of the graph: players
- Right side: actions with multiplicities according to x
- Edge to (all copies of) action if it is *potential* best response
- Claim: perfect matchings correspond to pure equilibria

Pure Nash Equilibria of Anonymous Games

$(0,1,1)$	$(0,0,1)$
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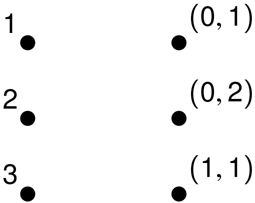

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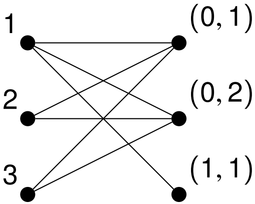

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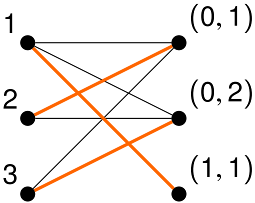

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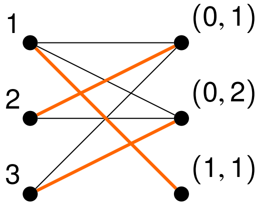

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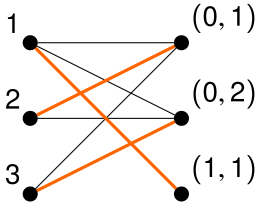
- As hard as finding matchings (*i.e.*, NL-hard)?



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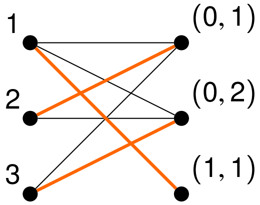

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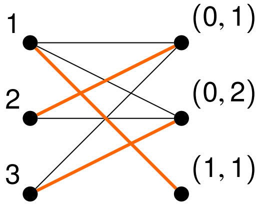

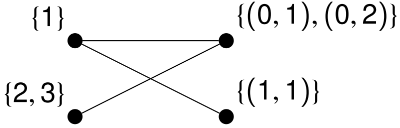

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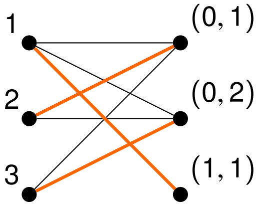
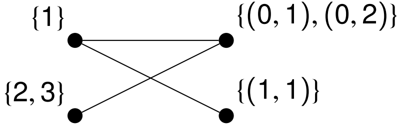
- Graph of constant size, condition can be checked in $\mathsf{TC}^0 \subseteq \mathsf{NL}$





We Have Only Just Begun

- Most interesting open problems:
  - quasi-strict equilibria of 2-player games
  - iterated weak dominance in 2-action anonymous games
- Quasi-strict equilibria and Shapley's saddles in restricted classes of games
- More restricted classes of games

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- Shapley's saddles: defined via inclusion-minimality, leads to interesting questions regarding complexity of search problem

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