Symmetries and the Complexity of Pure Nash Equilibrium

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Outline

Strategic Games

- Normal-form game $\Gamma = (N, (A_i)_{i \in N}, (p_i)_{i \in N})$
  - N a set of *players*
  - A_i a nonempty set of *actions* for player i
  - $p_i : (\times_{j \in N} A_j) \rightarrow \mathbb{R}$ a *payoff function* for player i
- Examples: Prisoners' Dilemma, Matching Pennies

	C	D
C	$(2,2)$	$(0,3)$
D	$(3,0)$	$(1,1)$

	H	T
H	$(1,0)$	$(0,1)$
T	$(0,1)$	$(1,0)$

- Strategy $s_i \in \Delta(A_i)$: probability distribution over A_i
- *Strategy profile* $s \in \times_{i \in N} \Delta(A_i)$
- *Pure* strategy: a degenerate distribution

Nash Equilibrium

- Informally: a profile of strategies that are *mutual best responses* to each other
- Formally: s is a *Nash equilibrium* if for every player $i \in N$, s_i is a *best response* to s_{-i}, i.e., for every $a \in A_i$,

$$p_i(s) \geq p_i((s_{-i}, a)),$$

  where $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ and $(s_{-i}, a) = (s_1, \ldots, s_{i-1}, a, s_{i+1}, \ldots, s_n)$
- General existence theorem (Nash, 1951): every finite game Γ has at least one equilibrium
- *Pure* Nash equilibrium: Nash equilibrium that is a pure strategy profile; *not* guaranteed to exist

Complexity of Nash Equilibrium

- PPAD complete for $|N| \geq 2$ by reduction from Brouwer fixed points (Daskalakis and Papadimitriou 2006; Chen and Deng, 2006)
- *Pure* Nash equilibria: existence decidable by enumeration of action profiles, complexity depends on *representation*
- List of payoffs for every action profile requires space $|N| \cdot |A|^{|N|}$
- *Succinct representations*
  - Congestion games (Rosenthal, 1973): PLS-complete (Fabrikant et al., 2004)
  - Graphical normal form (Kearns et al., 2001): NP-complete (Gottlob et al., 2005; Fischer et al., 2006)
  - Circuit form: NP-complete (Schoenebeck and Vadhan, 2006)

Outline

Symmetries and Succinct Representation

- Idea: Exploit similarities between players to enable succinct representation
- Prerequisite: $A_1 = \cdots = A_n = A$
- Weak symmetry: players cannot or need not distinguish between other players, i.e.,

$$p_i(s) = p_i(t) \qquad \text{for all } i \in N \text{ and all } s, t \in A^N \\ \text{with } s_i = t_i \text{ and } \#(s_{-i}) = \#(t_{-i})$$

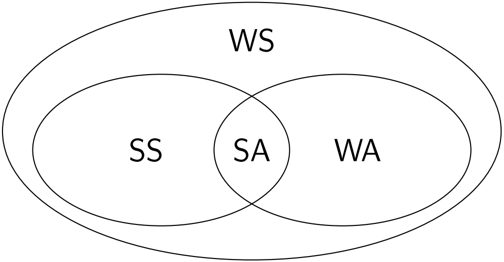
- $\#(s) = (\#(a, s))_{a \in A}$ is the commutative image (or Parikh image) of action profile s
- $\binom{n+k-1}{k-1}$ distributions of n players among k actions
- Representation has polynomial size *in general* if and only if k is a constant

Other Forms of Symmetry

- Strong symmetry: identical payoff functions for all players (in addition to the above), i.e.,

$$p_i(s) = p_j(t) \qquad \text{for all } i, j \in N \text{ and all } s, t \in A^N$$
$$\text{with } s_i = t_j \text{ and } \#(s_{-i}) = \#(t_{-j})$$

- Weak/strong anonymity: players do not distinguish *themselves* from the other players



Outline

Nash equilibria in symmetric games

- Every (strongly) symmetric game has a *symmetric* equilibrium (Nash, 1951)
- Symmetric equilibrium can be computed in P if $|A| = O(\log |N| / \log\log |N|)$ (Papadimitriou and Roughgarden, 2005)
- Does not apply to pure equilibria or weak symmetry
- Not obvious that symmetry simplifies the search for equilibria

(0, 1, 1)	(0,0,1)
(1,1,1)	(0, 0, 0)

(0,1,0)	(0, 0, 0)
(0, 1, 0)	(1, 0, 1)

Results

<table>
<tr><th></th><th>$|A| = O(1)$</th><th>$|A| = O(|N|)$</th></tr>
<tr><td>weakly symmetric</td><td rowspan="2">TC^0-complete</td><td rowspan="3">NP-complete</td></tr>
<tr><td>weakly anonymous</td></tr>
<tr><td>strongly symmetric</td><td rowspan="2">in AC^0</td></tr>
<tr><td>strongly anonymous</td><td>PLS-complete</td></tr>
</table>

- AC^0: Boolean circuits with constant depth, unbounded fan-in, polynomial size
- TC^0: AC^0 plus threshold gates
- $AC^0 \subset TC^0 \subseteq P \subseteq NP$
- PLS: polynomial local search

Weak Symmetry/Anonymity, $|A| = O(1)$

Membership in TC^0

- Fix a particular $x = \#(s)$, $s \in A^N$, and do the following:
  1. For each $C \subseteq A$, compute the number w_C of players for which C is the set of *potential* best responses under x
  2. Check whether the numbers $(w_C)_{C \subseteq A}$ are "compatible" with x
- Step 1 involves checking the Nash equilibrium condition
- Step 2 reduces to a homologous flow problem
- Constant $|A|$
  - Constant number of subsets C
  - x takes only polynomially many different values
- Certainly in P; membership in TC^0 can be shown by exploiting the structure of the flow network

Weak Symmetry/Anonymity, $|A| = O(1)$

TC^0-hardness

- Reduction from MAJORITY
- Game that has a pure Nash equilibrium iff *exactly* ℓ bits of an m-bit string are 1
- $m+2$ players of two different *types*
- Type of player i depends on value of ith input bit, players $m+1$ and $m+2$ are of of type 0 and 1, respectively
- Payoffs:

	0		...			$\ell+1$			...		$m+2$
p_0	...	0	1	0	2	1	0	1	0	1	...
p_1	...	1	0	1	0	1	2	0	1	0	...

Strong Symmetry/Anonymity, $|A| = O(1)$

- Unlike *weak* symmetry/anonymity: if s is a Nash equilibrium, so are all t with $\#(t) = \#(s)$
- We only need to check best response property for player *playing a certain action*, of which there are at most $|A|$
- Again, $\#(s)$, $s \in A^N$ takes only polynomially many different values
- Strongly anonymous games are common payoff; finding the maximum payoff (in AC^0) even finds a *social welfare maximizing* Nash equilibrium

Strong Symmetry/Weak Anonymity, $|A| = O(|N|)$

- Membership: Guess an action profile and verify that it is an equilibrium
- Hardness: reduction from satisfiability of a Boolean circuit $\mathcal{C}$ with inputs M (CSAT)
- Design game Γ with players $N = M$ and actions $A = \{ a_i^0, a_i^1 \mid i \in M \}$
- Action profile s corresponds to assignment of $\mathcal{C}$ if for every $i \in M$, $\#(a_i^0, s) + \#(a_i^1, s) = 1$
- Map *satisfying assignments* of $\mathcal{C}$ to Nash equilibria of Γ

Strong Anonymity, $|A| = O(|N|)$

- Strongly anonymous games are common payoff, always have a pure Nash equilibrium
- PLS: class of search problems where the existence of a solution is guaranteed by a local optimality argument
- Typical problem: finding a locally optimal solution of an NP-hard optimization problem
- Reduction from the PLS-complete problem FLIP to finding Nash equilibria in a weakly anonymous game with a growing number of actions and exponentially many payoffs

Outline

Conclusion

- Four notions of symmetry in multi-player games
- Finding pure Nash equilibria is tractable if the number of actions is a constant
- Identical payoff functions for all players simplify the problem
- A growing number of actions makes it intractable
- Anonymity seems to have no influence on the complexity
- Future work:
  - Games with a slowly growing number of actions
  - Mixed equilibria in weakly symmetric games
  - Player types, such that players of different types can be distinguished

Thank you for your attention!