On the Hardness and Existence of Quasi-Strict Equilibria

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1st International Symposium on Algorithmic Game Theory

Motivation

- Nash equilibrium: strategy profile that does not allow beneficial unilateral deviation
- Guaranteed to exist in finite games, but usually not unique
- *Equilibrium refinements* that single out particularly reasonable equilibria (see, *e.g.*, van Damme 1983)
- Quasi-strict equilibrium (Harsanyi, 1973): *every* best response played with positive probability
- Isolated q.s.e. are essential, strongly stable, regular, proper, and strictly perfect
- Satisfy Cubitt & Sugden (1994) axioms, existence of q.s.e. justifies assumption of common knowledge of rationality

Outline

Preliminaries

A Characterization in Matrix Games

Existence in Symmetric Games with Two Actions

NP-Hardness in Multi-Player Games

Quasi-Strict Equilibrium

- Normal-form game $\Gamma = (N, (A_i)_{i \in N}, (p_i)_{i \in N})$
  - N a set of *players*
  - A_i a nonempty set of *actions* for player i
  - $p_i : (\bigtimes_{j \in N} A_j) \to \mathbb{R}$ a *payoff function* for player i
- Nash equilibrium: strategy profile $s \in S = \bigtimes_{j \in N} \Delta(A_j)$ such that for all $i \in N$, $a \in A_i$,

$$p_i(s) \geq p(s_{-i}, a)$$

- Quasi-strict equilibrium: Nash equilibrium $s \in S$ such that for all $i \in N$ and $a, b \in A_i$ with $s_i(a) > 0$ and $s_i(b) = 0$,

$$p_i(s_{-i}, a) > p_i(s_{-i}, b)$$

Some Facts about Quasi-Strict Equilibria

- Guaranteed to exist in bimatrix games (Norde, 1999) and generic *n*-player games (Harsanyi, 1973)
- But not in three-player games, we will see an example later (others by van Damme, 1983; Kojima et al., 1984; Cubitt & Sugden, 1994; Brandt et al., 2007)

- PPAD-hard in bimatrix games (trivial)
- Membership in PPAD not obvious (Brouwer fixed point of a mapping that is complicated to construct)
- We will see it is likely harder in three-player games

A Characterization in Matrix Games

- **Theorem:** In matrix games, q.s.e. have a unique support, namely the set of all actions played in *some* Nash equilibrium
- LP characterization
  - Start from linear program for ordinary Nash equilibria
  - Primal and dual are feasible and have the same unique solution v (the "value" of the game)
  - Construct a feasibility program with the constraints of primal and dual, and additional constraints for $i \in \{0, 1\}$ and $a \in A_i$:

$$s_i(a) + v > \sum_{b \in A_{1-i}} s_{1-i}(b)\, p(a, b)$$

  - No additional restriction if $s_1(a) > 0$, but action a with $s_1(a) = 0$ yields payoff strictly less than v

Anonymous and Symmetric Games

- Anonymous game: payoff depends on own action and number of other players playing each of the different actions (but not their identities)
- Symmetric game: anonymous plus identical payoff functions for all players

- Observation: for symmetric matrix games the LP on the previous slide yields a symmetric equilibrium
- An anonymous game without quasi-strict equilibria:

$(1,1,0)$	$(0,1,1)$
$(0,1,1)$	$(1,0,1)$

$(0,1,1)$	$(1,0,1)$
$(1,0,1)$	$(1,1,0)$

Existence in Symmetric Games with Two Actions

- **Theorem:** Every symmetric game Γ with two actions has a quasi-strict equilibrium (not necessarily a symmetric one)
- Proof sketch:
  - Denote by p_{ma} the payoff from playing $a \in \{0, 1\}$ when m other players play action 1

p_{m0}	$\cdots$	$p_{\ell 0}$	$\cdots$	$p_{n-1,0}$
p_{m1}	$\cdots$	$p_{\ell 1}$	$\cdots$	$p_{n-1,1}$

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$$\begin{array}{|ccccc|} p_{m0} & \cdots & p_{\ell 0} & \cdots & p_{n-1,0} \\ & & & & \vee\!| \\ p_{m1} & \cdots & p_{\ell 1} & \cdots & p_{n-1,1} \end{array}$$

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		$\not\gtrless$		$\geq$
p_{m1}	$\cdots$	$p_{\ell 1}$	$\cdots$	$p_{n-1,1}$

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$=$		$\neq$		$\geq$
p_{m1}	$\cdots$	$p_{\ell 1}$	$\cdots$	$p_{n-1,1}$

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  - Payoffs have the form

$$\begin{array}{ccccccc} \cdots & p_{m-1,0} & p_{m0} & & p_{m'0} & p_{m'+1,0} & \cdots \\ & \wedge & \shortparallel & \cdots & \shortparallel & \vee & \\ \cdots & p_{m-1,1} & p_{m1} & & p_{m'1} & p_{m'+1,1} & \cdots \end{array}$$

  - Q.s.e. where $n - m' - 1$ players play 0, m players play 1, and $m' - m + 1$ players randomize

NP-Hardness in Multi-Player Games

- **Theorem:** Deciding whether a three-player game has a quasi-strict equilibrium is NP-complete
- Proof sketch:
  - Reduction from CLIQUE, inspired by McLennan & Tourky (2005)
  - Actions of players 1 and 2 correspond to vertices of a graph
  - Player 1 gets more payoff for vertices connected by an edge
  - Player 2 plays the same actions as player 1 in every equilibrium (imitation game)
  - Player 3 has two actions, with payoff the same as player 1 or depending on the desired clique size, respectively

NP-Hardness in Multi-Player Games

- **Theorem:** Deciding whether a three-player game has a quasi-strict equilibrium is NP-complete
- Proof sketch:

	b_1	$\cdots$	$b_{\lvert V\rvert}$	b_0
a_1				$(0,0,0)$
$\vdots$		$(m_{ij}, e_{ij}, m_{ij})_{i,j\in V}$		$\vdots$
$a_{\lvert V\rvert}$				$(0,0,0)$
a_0	$(0,0,0)$	$\cdots$	$(0,0,0)$	$(0,1,0)$

c_1

	b_1	$\cdots$	$b_{\lvert V\rvert}$	b_0
a_1	$(0,0,K)$	$\cdots$	$(0,0,K)$	$(0,0,0)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$a_{\lvert V\rvert}$	$(0,0,K)$	$\cdots$	$(0,0,K)$	$(0,0,0)$
a_0	$(1,0,0)$	$\cdots$	$(1,0,0)$	$(0,0,0)$

c_2

Conclusion

- Quasi-strict equilibrium: Nash equilibrium where *every* best response is played with positive probability
- Main results:
  - Every symmetric game with two actions has a quasi-strict equilibrium
  - Deciding existence in three-player games is NP-complete (so the search problem is NP-hard under Turing reductions)
- Open Problems:
  - Complexity of the search problem in bimatrix games
  - Existence in larger classes of multi-player games (*e.g.*, symmetric games with more than two actions)

Thank you for your attention!