The Price of Neutrality for the Ranked Pairs Method

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Social Choice

- Finite set A of alternatives
- Finite set $N = \{1, \ldots, n\}$ of voters, each with preferences over A
- Preference profile $R \in \mathcal{L}(A)^n$

  $\mathcal{L}(A)$: set of rankings of A (complete, transitive, asymmetric)
- $a\ R_i\ b$ means voter i strictly prefers a over b
- Social choice function (SCF) $f : \mathcal{L}(A)^n \to 2^A$
- Social preference function (SPF) $f : \mathcal{L}(A)^n \to 2^{\mathcal{L}(A)}$
- Central problem: $L \subseteq A \times A$ such that $a\ L\ b$ if and only if $|\{i \in N : a\ R_i\ b\}| > |\{i \in N : b\ R_i\ a\}|$ not necessarily transitive

Outline

Two Variants of the Ranked Pairs Method

Ranked Pairs Rankings, Winners, and Unique Winners

Possible and Necessary Ranked Pairs Winners

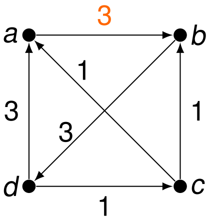
Ranked Pairs

- Insert elements into the social ranking by decreasing majority margin, while maintaining transitivity

  majority margin of a over b in R:

$$m_R(a,b) = |\{i \in N : a \; R_i \; b\}| - |\{i \in N : b \; R_i \; a\}|$$

2	1	3	1	2
a	*b*	*d*	*a*	*c*
b	*d*	*c*	*b*	*b*
c	*a*	*a*	*d*	*d*
d	*c*	*b*	*c*	*a*



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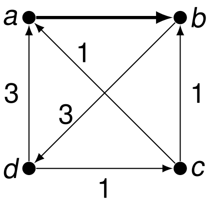

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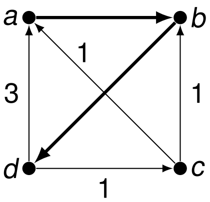

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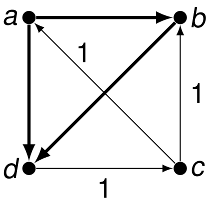

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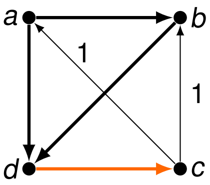

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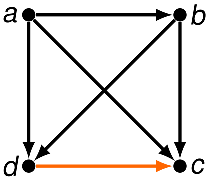

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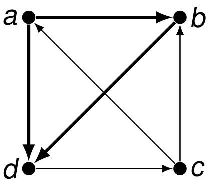

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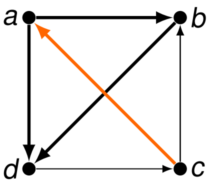

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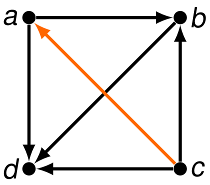

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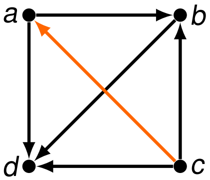

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- RPT(R,τ) for fixed *tie-breaking rule* $\tau \in \mathcal{L}(A \times A)$: resolute but not neutral
  - f is resolute if $|f(R)| = 1$ for every $R \in \mathcal{L}(A)^n$
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- $\text{RP}(R) = \bigcup_{\tau \in \mathcal{L}(A \times A)} \text{RPT}(R,\tau)$: neutral and irresolute, original definition of Tideman

Ranked Pairs Rankings

Finding a ranked pairs ranking is in P

- execute ranked pairs method for a specific tie-breaking rule

Deciding whether a given ranking is a ranked pairs ranking is in P

- Zavist, Tideman (1989): L is ranked pairs ranking iff L is stack
  - say *a attains b through L* if there are distinct $a_1, \dots, a_t$ such that $a_1 = a$, $a_t = b$, and for all $i = 1, \dots, t-1$,
    $$a_i \; L \; a_{i+1} \quad \text{and} \quad m_R(a_i, a_{i+1}) \geq m_R(b, a)$$
  - L is a *stack* if $a \; L \; b$ implies that a attains b through L
- deciding whether a ranking is a stack is in P
  - a attains b through L if there is a path from a to b in the directed graph $(A, \{(x, y) : x \; L \; y, m_R(x, y) \geq m_R(b, a)\})$

Ranked Pairs Winners

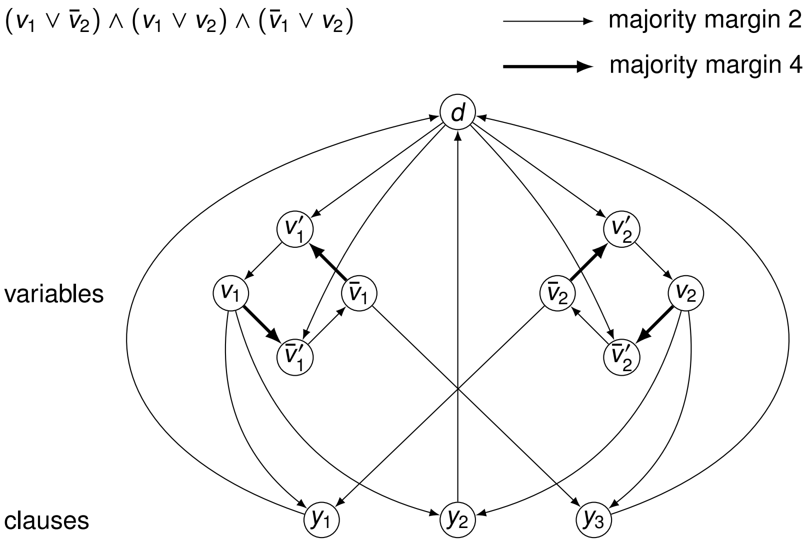
Finding a ranked pairs winner is in P

- execute ranked pairs method for a specific tie-breaking rule

Deciding whether a given alternative is a ranked pairs winner is NP-complete

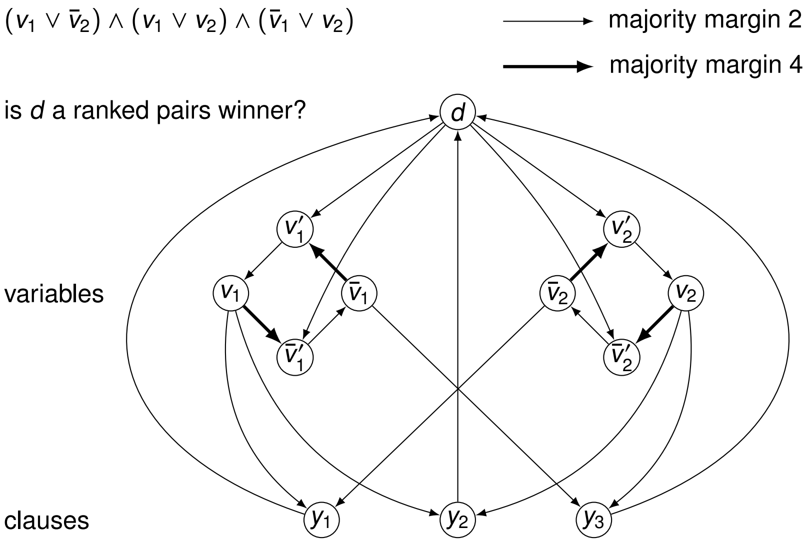
- membership: ranked pairs ranking with alternative at the top is a certificate
- hardness: reduction from SAT

$(v_1 \vee \bar{v}_2) \wedge (v_1 \vee v_2) \wedge (\bar{v}_1 \vee v_2)$



$(v_1 \vee \bar{v}_2) \wedge (v_1 \vee v_2) \wedge (\bar{v}_1 \vee v_2)$

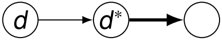
is d a ranked pairs winner?



Unique Winners

Deciding whether an alternative is the unique ranked pairs winner is coNP-complete

- membership: ranked pairs ranking with some other alternative at the top is a certificate
- hardness: extend NP-hardness construction above



  - d^* is unique ranked pairs winner iff formula is unsatisfiable
  - if it is satisfiable, d^* can be inserted in second position of ranked pairs ranking with d at the top

Possible and Necessary Ranked Pairs Winners

- Consider partially specified preference profile R: for each i, R_i is transitive and asymmetric, but not necessarily complete
- Preference profile R' is a completion of R if for all $i \in N$ and $a, b \in A$, $a\ R\ b$ implies $a\ R'\ b$
- Alternative a is a *possible* ranked pairs winner for R if it is a ranked pairs winner for *some* completion R' of R
- Alternative a is a *necessary* ranked pairs winner for R if it is a ranked pairs winner for *every* completion R' of R

New Proofs for Old and New Results

Deciding whether an alternative is a possible ranked pairs winner is NP-complete (Xia and Conitzer, 2011)

- completion and stack with alternative at the top is a certificate
- hardness: possible winner problem with complete preference profile is equivalent to winner problem

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- coNP-hardness: possible unique winner problem with complete preference profile is equivalent to unique winner problem

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Deciding whether an alternative is a possible unique ranked pairs winner is both NP-hard (Xia and Conitzer, 2011) and coNP-hard

- coNP-hardness: possible unique winner problem with complete preference profile is equivalent to unique winner problem

Necessary ranked pairs winner: coNP-hard and NP-hard
Necessary unique ranked pairs winner: coNP-complete

Summary

- Finding *some* ranked pair winner is easy
- Deciding whether given alternative is ranked pairs winner is hard
- Results for possible and necessary winner problems (some of them known) as corollaries
- Tradeoff between neutrality and tractability: RPT fails neutrality, RP is intractable
- Similar tradeoff for single transferrable vote (Conitzer et al., 2009; Wichmann, 2004)
- Ranked pairs easier *on average* than other intractable SCFs, ties unlikely to occur for most reasonable distributions of preferences

Non-Anonymous Variants

- Resoluteness and neutrality at the cost of anonymity

  f is anonymous if $f(\pi(R)) = \pi(f(R))$ for every $R \in \mathcal{L}(A)^n$ and every permutation π of N

- Use preferences of specific voter, or chairperson, to break ties

- A priori: use preferences of chairperson to define $\tau \in \mathcal{L}(A \times A)$

  efficiently computable

- A posteriori: choose $a \in \mathrm{RP}(R)$ most preferred by chairperson

  intractable

- Resoluteness, tractability, and appropriate generalizations of anonymity and neutrality by choosing chairperson at random

Thank you!