Mix and Match

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CRCS Seminar

Kidney Exchanges

- End Stage Renal Disease: fatal unless treated with dialysis or transplantation of a kidney
- Live donation possible, studies show no long-term negative effect on donor
- But: patients might not be compatible with their potential donor (usually a relative or friend)
- Kidney exchanges enable transplantations in such cases
- Basic case: two donor-patient pairs such that each donor has desired level of compatibility with patient of respective other pair

Incentives in Kidney Exchanges

- Incentives within a single (paired) exchange controlled by conducting transplantations simultaneously
- As kidney exchanges become more prevalent, incentives of hospitals also become an issue
- Several hospitals in which patients are treated
- Each hospital has an incentive to transplant its own patients
- Ideally: one large (regional or countrywide) exchange
- But: hospitals might want to “hide” some of their patients and carry out transplantations among them

Outline

A Model of Hospitals' Incentives

Lower Bounds

Deterministic Mechanisms

Mix and Match: A 2-Efficient Randomized Mechanism

A Model of Hospitals' Incentives (Roth et al., 2007)

- Set N of agents, corresponding to hospitals
- Graph $G = (V, E)$ with $V = \biguplus_i V_i$
  - Each vertex $v \in V$ corresponds to a donor-patient pair
  - Edge $(u, v) \in E$ means donor of u is compatible with patient of v, and donor of v with patient of u

- Agents report subsets $V_i' \subseteq V_i$
- Mechanism produces matching M' of subgraph induced by $\biguplus_i V_i'$
- Agent i adds matching $\widehat{M}_i$ for hidden and unmatched vertices
- Utility of agent i is number of vertices in V_i' matched in M' and $\widehat{M}_i$

Desirable Properties of Mechanisms

- Strategyproofness: no agent can gain by hiding vertices
- Approximate efficiency: mechanism is α-efficient if ratio between size of maximum cardinality matching and returned matching is at most α
- No monetary payments, for legal and ethical reasons

- "Approximate mechanism design without money" (Procaccia and Tennenholtz, 2009)

Lower Bounds

Theorem: If there are at least two agents,

no deterministic strategyproof mechanism can be α-efficient for $\alpha < 2$, and

no randomized strategyproof mechanism can be α-efficient for $\alpha < 4/3$.

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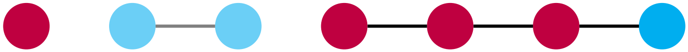

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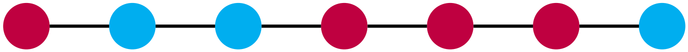

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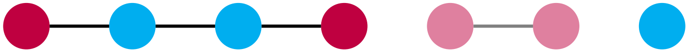

A Deterministic Mechanism for Two Agents

- Choose matching that has
  (i) maximum cardinality on both V_1 and V_2 and
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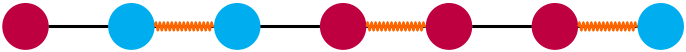

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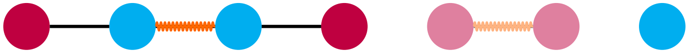

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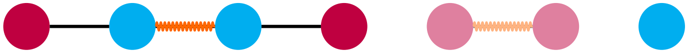

- Strategyproof (more on that later)
- 2-efficient, as returned matching is inclusion-maximal

A Generalization

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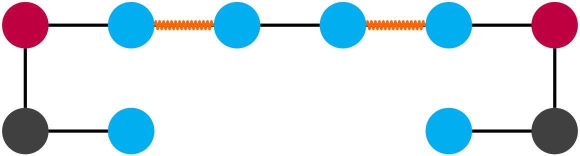

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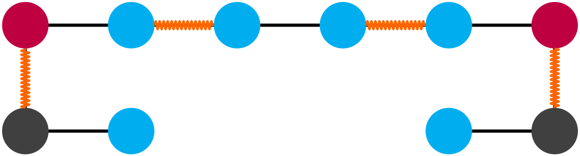

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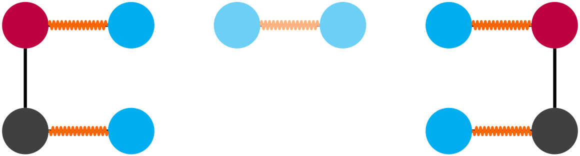

A Better Generalization

- Fix a bipartition $\Pi = (\Pi_1, \Pi_2)$ of N
- $\textsc{Match}_\Pi$: choose matching that has
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Theorem: For any number of agents and any bipartition Π, MATCH_Π is strategyproof.

Proof idea: . . .

Proof Idea

- Graph G, bipartition $\Pi = (\Pi_1, \Pi_2)$, $M = \textsc{Match}_\Pi(G)$
- Agent i hides vertices, $M' = \textsc{Match}_\Pi(G') \cup \widehat{M}_i$
- $M \Delta M' = (M \cup M') \setminus (M \cap M')$
  - vertex-disjoint paths, edges alternate between M and M'
  - cycles: have even length, both M and M' match all vertices
  - argument treats paths independently
- Thus assume w.l.o.g.: $M \Delta M'$ is a single path, not a cycle
- Arbitrarily fix a direction for this path
  - start and end vertex
  - (maximal) subpaths inside V_j, from V_i back to V_i
  - edges entering and leaving V_j

Proof Idea

- Two cases: $|M_{ii}| > |M'_{ii}|$ and $|M_{ii}| = |M'_{ii}|$, we consider the first
- Both M and M' have maximum cardinality on V_j for $j \neq i$
  $\Rightarrow$ every subpath inside V_j for $j \neq i$ has even length
  $\Rightarrow$ enters with M and leaves with M', or vice versa
- $M_{jk} = \emptyset$ when $j, k \in \Pi_\ell$ for $\ell \in \{1, 2\}$
  $\Rightarrow$ path crosses bipartition whenever it enters a new set
  $\Rightarrow$ leaving V_i with M it returns with M', and vice versa
- $(M \Delta M') \setminus (M_{ii} \cup M'_{ii})$
  - collection of subpaths
  - all but two of them have one edge in M_{ij} and one edge in M'_{ik} for some $j, k \neq i$

Proof Idea

$$u_i(M) = 2|M_{ii}| + \sum_{j \neq i} |M_{ij}|$$

$$2(|M'_{ii}| + 1) + \sum_{j \neq i} |M'_{ij}| - 2$$

$$= 2|M'_{ii}| + \sum_{j \neq i} |M'_{ij}| = u_i(M')$$

Proof Idea

$$
\begin{array}{rcccccc}
u_i(M) & = & 2|M_{ii}| & + & \sum_{j\neq i} |M_{ij}| & & \\
 & & \text{IV} & & & & \\
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- by the assumption that $|M_{ii}| > |M'_{ii}|$

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\begin{aligned}
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&\quad\ \ \text{IV} \qquad\qquad \text{IV} \\
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Theorem: For any number of agents and any bipartition Π, $\textsc{Match}_\Pi$ can be executed in polynomial time.

Proof idea: reduction to weighted matching

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- Strategyproof and 2-efficient mechanism for two agents
- No finite approximation ratio for more than two agents

Mix and Match

- MIX-AND-MATCH:
  1. Construct random bipartition $\Pi = (\Pi_1, \Pi_2)$: for each agent flip a fair coin to determine whether he goes to Π_1 or Π_2
  2. Execute MATCH_Π

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Theorem: For any number of agents, MIX-AND-MATCH is universally strategyproof and 2-efficient in expectation.

What We (Don't) Know

	deterministic		randomized	
	lower bound	upper bound	lower bound	upper bound
two agents	2	2	4/3	2
n agents	2	∞	4/3	2

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FLIP-AND-MATCH: with probability 1/2 each

- execute MIX-AND-MATCH
- return a maximum cardinality matching

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	lower bound	upper bound	lower bound	upper bound
two agents	2	2	$4/3$	2
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- Possible extensions
  - Stronger notion of stability: group-strategyproofness
  - Longer exchange sequences

Thank you!