Non-Truthful Position Auctions Are More Robust to Misspecification

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Mechanism Design

- Algorithm design with inputs held by strategic agents
- Example: allocation of an item to agent who values it the most
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- End of story?

Mechanism Design Theory and Practice

- Mechanism: solicit reports, choose allocation and payments
- Reports need not be truthful, analyze using game theory
- Revelation principle: can focus on direct truthful mechanisms
- VCG mechanisms (Vickrey, 1961; Clarke, 1971; Groves, 1973)
  - maximize social welfare relative to reports, charge externalities
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Position Auctions (Varian, 2007; Edelman et al., 2007)

- Set $\{1, \ldots, k\}$ of positions with relative values $\beta_1 \geq \cdots \geq \beta_k$
- Set $\{1, \ldots, n\}$ of bidders with unit demand and values $v_1, \ldots, v_n$
- Valuation of bidder i for position j is $\beta_j v_i$

- Auctioneer does not know $v_1, \ldots, v_n$
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- Real-world application: sponsored search
  - contributed over 90% to Google's revenue of \$66 billion in 2014
  - v_i: value bidder i has for a conversion
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- Mechanisms used in practice seem to defy theoretical prediction

Position Auctions (Varian, 2007; Edelman et al., 2007)

- Mechanism: allocation rule $g : \mathbb{R}^n \to S_n$

  payment rule $p : \mathbb{R}^n \to \mathbb{R}^n$

- Bidder i reports b_i to maximize $u_i(\mathbf{b}, v_i) = \beta_{g_i(\mathbf{b})} \cdot v_i - p_i(\mathbf{b})$

- Complete information: $\mathbf{v}$ common knowledge among bidders

  Nash equilibrium (NE): $\mathbf{b} \in \mathbb{R}^n$ such that for all i,

$$u_i(\mathbf{b}, v_i) = \max_{b_i' \in \mathbb{R}} u_i\big((b_i', \mathbf{b}_{-i}), v_i\big)$$

- Incomplete information: $\mathbf{v} \sim F$, where F is common knowledge

  Bayes-Nash equilibrium (BNE): $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ such that for all i, v_i,

$$\mathbb{E}_{\mathbf{v}_{-i} \sim F_{-i}|v_i}\Big[u_i\big(\mathbf{b}(v_i, \mathbf{v}_{-i}), v_i\big)\Big] = \max_{b_i' \in \mathbb{R}} \mathbb{E}_{\mathbf{v}_{-i} \sim F_{-i}|v_i}\Big[u_i\big((b_i', \mathbf{b}_{-i}(v_{-i})), v_i\big)\Big]$$

Mechanisms for Position Auctions

- Order bidders s.t. $b_1 \geq \cdots \geq b_n$, assign position i to bidder i
- Goal: efficient allocation, s.t. $v_1 \geq \cdots \geq v_n$
- Vickrey-Clarke-Groves (VCG) payments

$$p_i(\mathbf{b}) = \sum_{j=i}^{k}(\beta_j - \beta_{j+1})b_{j+1}$$

- Generalized First-Price (GFP) payments $p_i(\mathbf{b}) = \beta_i b_i$
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- First successful sponsored search service, Overture, used GFP
- Google and Bing use GSP

Why Not Not Non-Truthful Mechanisms?

	VCG	GFP	GSP
NE	truthful	possibly none	efficient
BNE	truthful	efficient	possibly none

- GSP mechanism
  - NE producing same outcome as truthful NE of VCG mechanism (Varian, 2007; Edelman et al., 2007)
  - any NE at least 0.78-efficient (Caragiannis et al., 2015)
  - possibly no BNE (Gomes and Sweeney, 2014)
- GFP mechanism
  - unique BNE, which is efficient (Chawla and Hartline, 2013)
  - possibly no NE (Varian, 2007; Edelman et al., 2007)
- Why not the truthful VCG mechanism?
  - may have low revenue in an efficient NE (Milgrom, 2010)

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- Conversions may in fact not be observable (Milgrom, 2010)
- Advertisers can report conversions but may choose not to
- Search engines learn from data, likely to be close but not exact

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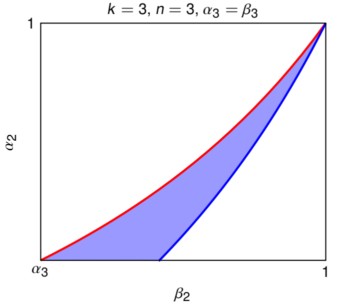
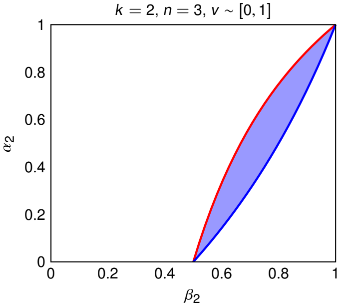
- Robustness to misspecification: for which values of α and β does each of the mechanisms possess an efficient equilibrium?

Results

Theorem 1: Under complete information, the GSP mechanism possesses an efficient equilibrium for a strictly larger set of values of α and β than the VCG mechanism.

Theorem 2: Under incomplete information, when *F* is symmetric, the GFP mechanism possesses an efficient equilibrium for a strictly larger set of values of α and β than the VCG mechanism.





Robustness and Simplicity

- This talk: efficient equilibrium for given α and β
- Dütting, F, Parkes, 2011, 2014: efficient equilibrium for every β
  - complete information: GSP mechanism with $\alpha = (1, \ldots, 1)$
  - incomplete information: GFP mechanism with $\alpha = (1, \ldots, 1)$
  - complete and incomplete information: GFP mechanism with multi-dimensional bids, and these are necessary

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- Non-truthful mechanisms are consistently more robust
- Intuitively a consequence of simpler payments

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- Non-truthful mechanisms are consistently more robust
- Intuitively a consequence of simpler payments
- Do results extend to more general settings?
- Challenge: pushes the boundary of equilibrium analysis
- Can equilibrium bids be verified empirically?
- Challenge: data not available, not in the public domain

Simplicity and Robustness

- What makes mechanisms robust to misspecification?
- Box (1976): “all models are wrong”
- Any scientific model of the real world is a simplification
- Important property is not correctness, but usefulness
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- Simplicity may improve usability and may itself be desirable
- Not a question that is commonly addressed in mechanism design
- Madarász and Prat (2017): results for the screening problem

Complete Information: $k = 3$, $n = 3$

- Assume $v_1 \geq v_2 \geq v_3$, efficiency requires $b_1 \geq b_2 \geq b_3$
- Equilibrium conditions for VCG mechanism:

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- Efficient equilibrium if and only if there exists b_2 such that

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- When $\alpha_3 > 0$, a strict separation exists between the mechanisms

$$\text{VCG:}\quad (\alpha_1 - \alpha_2)b_2 \leq (\beta_1 - \beta_2)v_1, \quad (\alpha_2 - \alpha_3)b_2 \geq (\beta_2 - \beta_3)v_3$$

$$\text{GSP:}\quad (\alpha_1 - \alpha_2)b_2 \leq (\beta_1 - \beta_2)v_1, \quad \alpha_2 b_2 \geq (\beta_2 - \beta_3)v_3$$

- $\alpha_3 > 0$ necessary for equilibrium existence when $n > 3$
- To illustrate the separation, assume $\beta_1 > \beta_2 > \beta_3$, $\alpha_1 > \alpha_2 > \alpha_3$

$$\text{VCG:}\quad \alpha_2 \geq \frac{\alpha_1(\beta_2-\beta_3)v_3+\alpha_3(\beta_1-\beta_2)v_1}{(\beta_1-\beta_2)v_1+(\beta_2-\beta_3)v_3}$$

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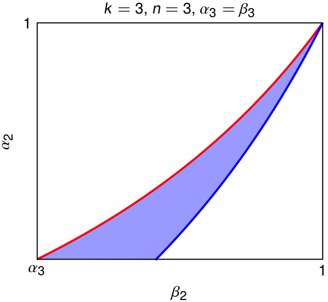

Complete Information: General Case

- Direct analysis tedious or infeasible
- Show instead: for VCG mechanism, existence of efficient Nash equilibrium implies existence of efficient envy-free equilibrium
- Bid vector b is envy-free if for all $i \in \{1, \ldots, n\}$,

$$\beta_{g_i(b)} v_i - p_i(b) = \max_{j \in \{1,\ldots,n\}} \beta_{g_j(b)} v_i - p_j(b)$$

- For both mechanisms: stronger than Nash equilibrium condition
- Can be viewed as property of allocation and payments rather than of underlying mechanism
- To complete the proof: map bids in VCG mechanism to bids in GSP mechanism producing same allocation and payments

Incomplete Information: Overview

- Efficiency requires symmetric bidding functions

Lemma (Myerson,1981): Assume bidders use symmetric bidding function b, and a bidder with value v is consequently assigned position $s \in \{1, \dots, k\}$ with probability $P_s(v)$. Then b is a BNE if and only if the following holds:

(a). the expected allocation $\sum_{s=1}^{k} P_s(v)\beta_s$ is non-decreasing in v,

(b). the (symmetric) payment function p satisfies

$$\mathbb{E}[p(v)] = p(0) + \sum_{s=1}^{k} \beta_s \int_0^v \frac{dP_s(z)}{dz} z \, dz.$$

- Find candidate equilibrium bidding function for a mechanism by equating its payment rule with expression in Myerson's Lemma
- An actual equilibrium if and only if increasing almost everywhere
- Show that this holds for GFP whenever it holds for VCG

Incomplete Information: $k = 2$, $n = 3$, $v \sim U[0, 1]$

$$P_1(v) = F^2(v) = v^2 \qquad P_2(v) = \binom{2}{1} F(v)(1 - F(v)) = 2v(1 - v)$$

$$\mathbb{E}[p(v)] = \sum_{m=1}^{2} \beta_m \int_0^v \frac{dP_s(z)}{dz} z\, dz = \tfrac{2}{3}\beta_1 v^3 + \beta_2 v^2 - \tfrac{4}{3}\beta_2 v^3$$

$$\begin{aligned}\mathbb{E}[p^G(v)] &= P_1(v)\alpha_1 b^G(v) + P_2(v)\alpha_2 b^G(v)\\ &= (\alpha_1 v^2 + 2\alpha_2 v - 2\alpha_2 v^2) b^G(v)\end{aligned}$$

$$b^G(v) = \frac{2/3 \cdot v^3 - 4/3 \cdot \beta_2 v^2 + \beta_2 v^2}{v^2 - 2\alpha_2 v^2 + 2\alpha_2 v}$$

$$\frac{db^G(v)}{dv} = \frac{(\frac{4}{3} v - \frac{8}{3}\beta_2 v + \beta_2)(v - 2\alpha_2 v + 2\alpha_2) - (1 - 2\alpha_2)(\frac{2}{3} v^2 - \frac{4}{3}\beta_2 v^2 + \beta_2 v)}{(v - 2\alpha_2 v + 2\alpha_2)^2}$$

Increasing almost everywhere if and only if $\beta_2 \leq \frac{1}{2}$ or $\alpha_2 \geq \frac{2\beta_2 - 1}{2 - \beta_2}$

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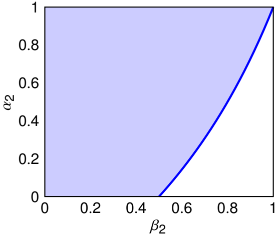
$$P_1(v) = F^2(v) = v^2 \qquad P_2(v) = \binom{2}{1}F(v)(1 - F(v)) = 2v(1 - v)$$

$$\mathbb{E}[p(v)] = \sum_{m=1}^{2} \beta_m \int_0^v \frac{dP_s(z)}{dz} z\, dz = \tfrac{2}{3}\beta_1 v^3 + \beta_2 v^2 - \tfrac{4}{3}\beta_2 v^3$$

$$\begin{aligned}\mathbb{E}[p^G(v)] &= P_1(v)\alpha_1 b^G(v) + P_2(v)\alpha_2 b^G(v)\\ &= (\alpha_1 v^2 + 2\alpha_2 v - 2\alpha_2 v^2) b^G(v)\end{aligned}$$



$$b^G(v) = \frac{2/3 \cdot v^3 - 4/3 \cdot \beta_2 v^2 + \beta_2 v^2}{v^2 - 2\alpha_2 v^2 + 2\alpha_2 v}$$

$$\frac{db^G(v)}{dv} = \frac{(\frac{4}{3}v - \frac{8}{3}\beta_2 v + \beta_2)(v - 2\alpha_2 v + 2\alpha_2) - (1 - 2\alpha_2)(\frac{2}{3}v^2 - \frac{4}{3}\beta_2 v^2 + \beta_2 v)}{(v - 2\alpha_2 v + 2\alpha_2)^2}$$

Increasing almost everywhere if and only if $\beta_2 \leq \frac{1}{2}$ or $\alpha_2 \geq \frac{2\beta_2 - 1}{2 - \beta_2}$

Incomplete Information: $k = 2$, $n = 3$, $v \sim U[0, 1]$

$$
\begin{aligned}
\mathbb{E}[p^V(v)] &= P_1(v)\left[(\alpha_1 - \alpha_2)\int_0^v \tfrac{2t}{v^2} b^V(t)\,dt + \alpha_2 \int_0^v \tfrac{2(v-t)}{v^2} b^V(t)\,dt\right] + \\
&\quad P_2(v)\alpha_2 \int_0^v \tfrac{1}{v} b^V(t)\,dt \\
&= (2\alpha_1 - 4\alpha_2)\int_0^v t b^V(t)\,dt + \\
&\quad 2\alpha_2 \int_0^v b^V(t)\,dt,
\end{aligned}
$$

$\frac{2t}{v^2} = \frac{2F(t)f(t)}{F(v)^2}$: density of second-highest value given v is highest

$\frac{2(v-t)}{v^2} = \frac{2F(v-t)f(t)}{F(v)^2}$: density of third-highest value given v is highest

$\frac{1}{v} = \frac{f(t)}{F(v)}$: density of third-highest value given v is second highest

Incomplete Information: $k = 2$, $n = 3$, $v \sim U[0, 1]$

$$\mathbb{E}[p^V(v)] = P_1(v)\left[(\alpha_1 - \alpha_2)\int_0^v \tfrac{2t}{v^2} b^V(t)\,dt + \alpha_2 \int_0^v \tfrac{2(v-t)}{v^2} b^V(t)\,dt\right] +$$
$$P_2(v)\alpha_2 \int_0^v \tfrac{1}{v} b^V(t)\,dt$$
$$= (2\alpha_1 - 4\alpha_2)\int_0^v t b^V(t)\,dt +$$
$$2\alpha_2 \int_0^v b^V(t)\,dt,$$

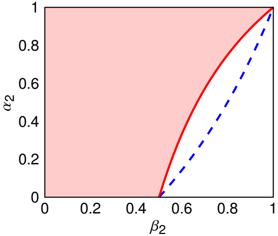
$$b^V(v) = \tfrac{2v^2 - 4\beta_2 v^2 + 2\beta_2 v}{2v - 4\alpha_2 v + 2\alpha_2}$$

$$\frac{db^V(v)}{dv} = \frac{(4v - 8\beta_2 v + 2\beta_2)(2v - 4\alpha_2 v + 2\alpha_2) - (2 - 4\alpha_2)(2v^2 - 4\beta_2 v^2 + 2\beta_2 v)}{(2v - 4\alpha_2 v + 2\alpha_2)^2}$$

Increasing almost everywhere if and only if $\beta_2 \leq \frac{1}{2}$ or $\alpha_2 \geq 2 - \frac{1}{\beta_2}$

Incomplete Information: $k = 2$, $n = 3$, $v \sim U[0,1]$

$$\begin{aligned}\mathbb{E}[p^V(v)] &= P_1(v)\left[(\alpha_1 - \alpha_2)\int_0^v \frac{2t}{v^2} b^V(t)\,dt + \alpha_2 \int_0^v \frac{2(v-t)}{v^2} b^V(t)\,dt\right] + \\ &\quad P_2(v)\alpha_2 \int_0^v \frac{1}{v} b^V(t)\,dt \\ &= (2\alpha_1 - 4\alpha_2)\int_0^v t b^V(t)\,dt + \\ &\quad 2\alpha_2 \int_0^v b^V(t)\,dt,\end{aligned}$$



$$b^V(v) = \frac{2v^2 - 4\beta_2 v^2 + 2\beta_2 v}{2v - 4\alpha_2 v + 2\alpha_2}$$

$$\frac{db^V(v)}{dv} = \frac{(4v - 8\beta_2 v + 2\beta_2)(2v - 4\alpha_2 v + 2\alpha_2) - (2 - 4\alpha_2)(2v^2 - 4\beta_2 v^2 + 2\beta_2 v)}{(2v - 4\alpha_2 v + 2\alpha_2)^2}$$

Increasing almost everywhere if and only if $\beta_2 \leq \frac{1}{2}$ or $\alpha_2 \geq 2 - \frac{1}{\beta_2}$

Incomplete Information: General Case

- Bidding function for VCG mechanism leads to ODE, solve using theory of order statistics of i.i.d. random variables, plus magic

$$b^G(v) = \frac{\sum_{s=1}^k \beta_s \int_0^v \frac{dP_s(t)}{dt} t \, dt}{\sum_{s=1}^k \alpha_s P_s(v)} \qquad b^V(v) = \frac{\sum_{s=1}^k \beta_s \frac{dP_s(v)}{dv} v}{\sum_{s=1}^k \alpha_s \frac{dP_s(v)}{dv}}$$

- In absence of closed-form expressions, show that $b^G(v)$ is increasing whenever $b^V(v)$ is by observing that

$$b^G(v) = \frac{A(v)}{B(v)} \quad \text{and} \quad b^V(v) = \frac{A'(v)}{B'(v)}$$

for appropriately chosen $A : \mathbb{R} \to \mathbb{R}$ and $B : \mathbb{R} \to \mathbb{R}$

Incomplete Information: General Case

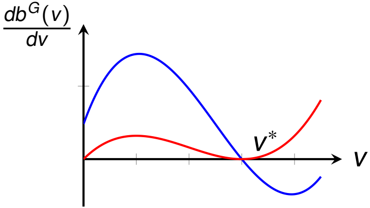
- b^G starts out non-decreasing:

$$\frac{db^G(v)}{dv}\bigg|_{v=0} > 0, \quad \text{or} \quad \frac{db^G(v)}{dv}\bigg|_{v=0} = 0 \quad \text{and} \quad \frac{d^2 b^G(v)}{dv^2}\bigg|_{v=0} \geq 0$$

- Equilibrium existence for GFP depends on values $v^* > 0$ with

$$\frac{db^G(v)}{dv}\bigg|_{v=v^*} = 0$$

- Show that at any such point, b^V behaves in roughly the same way



Thank you!