# Optimal Impartial Selection

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(joint work with Max Klimm, TU Berlin)

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# Impartial Selection

- Select member of a set of agents based on nominations by agents from the same set
- Applications
  - selection of representatives
  - award of a prize
  - assignment of responsibilities
  - peer review: papers, research proposals, . . .
- Assumption: agents are impartial to the selection of *other* agents
  - will reveal their opinion truthfully. . .
  - as long as it does not affect their own chance of selection
- Goal: preserve impartiality, select agent with many nominations

# A Formal Model

- Set $\mathcal{G}$ of graphs $(N, E)$ without self loops

  vertices represent agents, $(i, j) \in E$ means $i$ nominates $j$
- $\delta^-_S(i, G) = \left|\left\{(j, i) \in E : G = (N, E), j \in S\right\}\right|$

  number of nominations $i \in N$ receives from $S \subseteq N$

- selection mechanism: maps each $G \in \mathcal{G}$ to distribution on $N$
- $f$ is impartial if

$$\big(f((N, E))\big)_i = \big(f((N, E'))\big)_i \quad \text{if} \quad E \setminus (\{i\} \times V) = E' \setminus (\{i\} \times V)$$

- $f$ is $\alpha$-optimal, for $\alpha \leq 1$, if for all $G \in \mathcal{G}$,

$$\frac{\mathbb{E}_{i \sim f(G)}[\delta^-_N(i, G)]}{\Delta(G)} \geq \alpha,$$

  where $\Delta(G) = \max_{i \in N} \delta^-_N(i, G)$

# Related Work

- Impartial Nominations for a Prize (Moulin, Holzman)
  - plurality, deterministic mechanisms, axiomatic study
- Strategyproof Selection from the Selectors (Alon et al.)
  - approval, deterministic and randomized mechanisms, selection of $k$ agents with large number of nominations
- Impartial Division of a Dollar (de Clippel et al.)
  - more general than randomized mechanisms, axiomatic study
- Plurality: one nomination per agent (outdegree one)
- Approval: zero or more nominations (arbitrary outdegree)

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|---|---|---|
| deterministic | 0 | $1/n$ |
| randomized | $[1/4, 1/2]$ | $[1/4, 1]$ |

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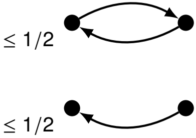

# Outline and Results

- 1/2-optimal mechanism for approval
- same mechanism is 7/12-optimal for plurality
  (may actually be 2/3-optimal, but not better)
- upper bound for plurality of roughly 3/4
- Lower bounds from
  - better analysis of the mechanism of Alon et al.
  - generalization of the analysis to a (fairly) natural generalization of the mechanism
- Upper bound from optimization approach to finding mechanisms

# The 2-Partition Mechanism (Alon et al.)

- Randomly partition $N$ into $(S_1, S_2)$
- Select $i \in \arg\max_{i' \in S_2} \delta^-_{S_1}(i', G)$ uniformly at random
- 1/4-optimal
  - consider any $G \in \mathcal{G}$ and vertex $i^*$ with degree $\Delta = \Delta(G)$
  - $i^* \in S_2$ with probability $1/2$
  - $\mathbb{E}[\delta^-_{S_1}(i, G) \mid i^* \in S_2] = \Delta/2$
  - vertex selected when $i^* \in S_2$ has at least this degree
- Analysis of the mechanism is tight (graph with a single edge)
- Analysis of the problem is not tight: not obvious how to close the gap between $1/4$ and $1/2$, and by which analysis

# The 2-Partition Mechanism (Revisited)

- Consider vertex $i^*$ with degree $\Delta$
- Randomly partition $N \setminus \{i^*\}$ into $(S_1, S_2)$
- Based on $(S_1, S_2)$ adversary chooses $d = \max_{i \in S_2} \delta^-_{S_1}(i, G)$
- $i^*$ goes to $S_1$ or $S_2$ with probability $1/2$ each
- Depending on $d^* = \delta^-_{S_1}(i^*, G)$, adversary will
  - set $d$ to 0 and let $i^*$ win with probability $1/2$
  - set $d$ to $d^*$ and beat $i^*$ (assume ties broken against $i^*$)
- Selected vertex has expected degree $\min\{\Delta/2, d^*\}$
- Sum over distribution of $d^*$

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- Parameterized lower bound $\alpha(\Delta)$ in closed form
  - non-decreasing in $\Delta$
  - $\alpha(1) = 1/4$
  - $\alpha(2) = 3/8$

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# The $k$-Partition Mechanism

randomly partition $N$ into $(S_1, \ldots, S_k)$, denote $S_{<j} = \bigcup_{i<j} S_i$

$\{i^*\} := \emptyset$, $d^* := 0$

for $j = 2, \ldots, k$

  if $\max_{i \in S_j} \delta^-_{S_{<j} \setminus \{i^*\}}(i, G) \geq d^*$

    choose $i \in \arg\max_{i' \in S_j} \delta^-_{S_{<j}}(i', G)$ uniformly at random

    $i^* := i$, $d^* := \delta^-_{S_{<j}}(i, G)$

select $i^*$

- Goal: parameterized lower bound $\alpha_k(\Delta)$

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- Goal: parameterized lower bound $\alpha_k(\Delta)$

# The $k$-Partition Mechanism

- Consider vertex $i^*$ with degree $\Delta$
- Randomly partition $N \setminus \{i^*\}$ into $(S_1, \ldots, S_k)$
- For $j = 2, \ldots, k$, adversary decides to beat $i^*$ or let it win if $i^* \in S_j$
- $i^*$ goes to each $S_j$ with probability $1/k$
- Only rightmost alternative to beat $i^*$ matters, as either that alternative or $i^*$ is selected
- For fixed $(S_1, \ldots, S_k)$, selected vertex has expected degree

$$\min_{j=1,\ldots,k} \left\{ \delta^-_{S_{<j}}(i^*, G) + \frac{k-j}{k}\left(\Delta - \delta^-_{S_{<j}}(i^*, G)\right) \right\}$$

- Sum over distribution of $\left(\delta^-_{S_j}(i^*, G)\right)_{j=1,\ldots,k}$

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    $i^* := i$, $d^* := \delta^-_{S_{<j}}(i, G)$

select $i^*$

- Parameterized lower bound $\alpha_k(\Delta)$
  - for every $k \geq 2$, non-decreasing in $\Delta = \Delta(G)$
  - $\alpha_k(1) \geq (k-1)/(2k)$
  - $\alpha_k(2) \geq 7/12 - o(k)/k$

# The Permutation Mechanism

pick random permutation $(\pi_1, \ldots, \pi_n)$ of $N$, denote $\pi_{<j} = \bigcup_{i<j} \{\pi_i\}$

$i^* := \pi_1$, $d^* := 0$

for $j = 2, \ldots, k$

  if $\delta^-_{\pi_{<j} \setminus \{i^*\}}(\pi_j) \geq d^*$

    $i^* := \pi_j$, $d^* := \delta^-_{\pi_{<j}}(\pi_j)$

return $i^*$

- Limit of $k$-partition mechanism as $k \to \infty$
- 1/2-optimal for approval, 7/12-optimal for plurality
- $k$-partition for fixed $k$ may be more desirable, allows more anonymous processing of ballots

# Upper Bound for Plurality

- For any $\alpha$-optimal impartial selection mechanism for plurality,

$$\alpha \leq \begin{cases} 5/6 & \text{if } n = 3, \\ (6n-1)/8n & \text{if } n \geq 6 \text{ even, and} \\ 3/4 & \text{otherwise} \end{cases}$$

# Upper Bound for Plurality

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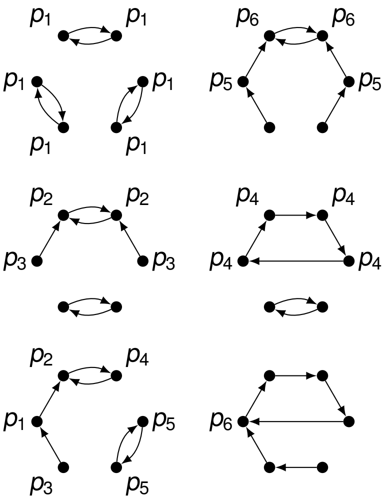
- Optimal mechanisms from LP
- Large number of constraints (linear in number of graphs)
- Upper bound for small graphs from dual, then generalize

# Upper Bound for Plurality, $n \geq 6$ even




W.l.o.g., only consider symmetric mechanisms

$$np_1 = 1$$
$$2p_2 + 2p_3 \leq 1$$
$$p_1 + p_2 + p_3 + p_4 + (n-4)p_5 = 1$$
$$2p_5 + 2p_6 \leq 1$$
$$4p_4 \leq 1$$

$$p_6 \leq 1/2 - 1/(4n)$$

$$\alpha \leq \frac{2p_6 + (1-p_6)}{2} = \frac{p_6 + 1}{2}$$

# Thank you!