OPTIMAL IMPARTIAL SELECTION*

FELIX FISCHER† AND MAX KLIMM‡

Abstract. We study a fundamental problem in social choice theory, the selection of a member of a set of agents based on impartial nominations by agents from that set. Studied previously by Alon et al. [Proceedings of TARK, 2011] and by Holzman and Moulin [Econometrica, 2013], this problem arises when representatives are selected from within a group or when publishing or funding decisions are made based on a process of peer review. Our main result concerns a randomized mechanism that in expectation selects an agent with at least half the maximum number of nominations. This is best possible subject to impartiality and resolves a conjecture of Alon et al. Further results are given for the case where some agent receives many nominations and the case where each agent casts at least one nomination.

Key words. Impartial Selection, Impartiality, Voting, Social Choice, Approximation

AMS subject classifications. 91A43, 91B12, 91B14, 68W25.

1. Introduction.

Consider a situation where members of a set of agents nominate other agents from the set for a prize and the goal is to award the prize to an agent who receives a large number of nominations. This situation arises naturally, for example, when representatives are selected from within a group or when publishing or funding decisions are made based on a process of peer review. While nominations are at the discretion of the nominating agents, it is often reasonable to assume that agents are impartial to the selection of others and will nominate who they think should receive the prize as long as they cannot influence their own chances of receiving it. Indeed, the assumption of impartiality was previously made, and justified, in the very same setting [2, 15].

Formally, the situation can be captured by a directed graph with n vertices, one for each agent, in which edges correspond to nominations. A selection mechanism then chooses a vertex for any given graph, and impartiality requires that the chances of a particular vertex to be chosen do not depend on its outgoing edges. It is easy to see that an impartial mechanism cannot always select a vertex with maximum indegree, corresponding to an agent with a maximum number of nominations, even when $n = 2$. We therefore aim at maximizing the indegree of the selected vertex relative to the maximum indegree, and call a mechanism α-optimal, for $\alpha \leq 1$, if for every graph the former is at least α times the latter. We focus here on the selection of a single agent, but note that it is an interesting question whether optimal mechanisms for selecting any given number of agents can be obtained directly from mechanisms for selecting a single agent, or whether their design requires additional techniques.

1.1. State of the Art.

Alon et al. [2] and Holzman and Moulin [15] showed independently that deterministic impartial mechanisms are extremely limited, and must sometimes select an agent with zero nominations even when agents are being

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†Statistical Laboratory, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK, fischerf@statslab.cam.ac.uk.

‡Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany, klimm@math.tu-berlin.de.

nominated, or an agent with one nomination when another agent receives $n-1$ nominations.

On the other hand, Alon et al. proposed a simple mechanism that randomly partitions the agents into two sets S_1 and S_2 and selects an agent from S_2 who among agents in this set receives a maximum number of nominations by agents in S_1. By linearity of expectation the mechanism is at least 1/4-optimal, and a situation with a single nomination shows that it cannot do better. A somewhat closer inspection of situations with one or two nominations shows that no impartial mechanism can be better than 1/2-optimal. While these bounds are almost trivial, no improvements have been obtained that hold for general values of n, despite considerable efforts. Improving the lower bound in fact appears just as difficult for the special case where each agent submits exactly one nomination, as considered by Holzman and Moulin. This is somewhat embarrassing, as the mechanism of Alon et al. should intuitively be better than 1/4-optimal as soon as there is more than just a single nomination.

1.2. Our Contribution. Alon et al.'s analysis of the 2-partition mechanism is tight and yields a constant approximation ratio of 1/4, only a factor of two away from the best possible one. Quite strikingly, however, the analysis does not reveal much of the structure of the problem. It does not lead to stronger bounds for special cases, like the setting with one nomination per agent studied by Holzman and Moulin, and cannot be extended to more complicated mechanisms.

We attempt to close this gap in our understanding of the 2-partition mechanism in Section 3, by providing a lower bound on its performance relative to the maximum indegree. We show that the performance of the mechanism increases monotonically with the maximum indegree and converges to 1/2. As a direct consequence we obtain a lower bound of 3/8 for the case where each agent submits at least one nomination. Our analysis uses a novel adversarial argument that allows us to abstract from the underlying graph structure and isolate the critical aspects of difficult problem instances.

More interestingly, the analysis extends to a natural generalization of the 2-partition mechanism, which we discuss in Section 4. This mechanism partitions the set of agents into $k > 2$ sets and iteratively considers the nominations submitted by agents in more and more of these sets, to fewer and fewer candidates in the remaining sets. Intuitively this increases the probability of each individual nomination to be counted, which is particularly important in the difficult cases with a small overall number of nominations. Exactly how information from an earlier stage of the mechanism can be used without a negative effect on later stages turns out to be somewhat intricate. A generalization of the adversarial analysis shows that the k-partition mechanism is $\frac{k-1}{2k}$-optimal, which approaches the upper bound of 1/2 as k tends to infinity. This implicitly provides an analysis of a limiting mechanism, discussed in Section 5, in which agents are considered one by one according to a random permutation. The existence of a 1/2-optimal impartial mechanism was in fact conjectured by Alon et al. [2].

In Section 7 we finally give the first non-trivial bounds for settings without abstentions, where each agent is required to cast at least one nomination. We show that the permutation mechanism is at least 67/108-optimal and at most 2/3-optimal in this case, and that no impartial mechanism can be more than 3/4-optimal.

Following the initial publication of our results [12], Bousquet et al. [4] studied the asymptotic performance of impartial mechanisms as the maximum number of nominations for any agent tends to infinity. They showed in particular that the permutation mechanism is $(3/4-\epsilon)$-optimal in this case, and proposed a new mechanism that is

$(1-\epsilon)$-optimal. The proof of the former result uses the notion of a balanced permutation and is quite technical, but we will see in Section 6 that it can also be obtained from our results by a straightforward application of Chebychev's inequality. Neither result provides an exact bound relative to the maximum number of nominations, and such bounds may be rather difficult to obtain.

1.3. Related Work and Applications. Impartial decision making was first considered by de Clippel et al. [6], for the case of a divisible resource to be shared among a set of agents. While the difference between a divisible resource and the indivisible resource considered in this article disappears for randomized mechanisms, de Clippel et al. studied mechanisms with a more general message space that allows for fractional nominations and at the same time aimed for a different set of requirements to be achieved besides impartiality. Their results thus do not have any obvious consequences for our setting.

Alon et al. [2] framed the problem considered here as one of designing approximately optimal strategyproof mechanisms without payments, an agenda proposed by Procaccia and Tennenholtz [17] and earlier by Dekel et al. [7]. Strategyproofness requires that an agent maximizes its utility by truthfully revealing its preferences and is equivalent to impartiality if the utility of an agent only depends on its chances of being selected. While this assumption seems somewhat restrictive, Alon et al. pointed out that their results in fact hold for any setting where agents give their own selection priority over that of their nominees. The same is true for our results as well.

Strategyproof selection is an important component of the peer review process for scientific articles and project proposals. For its Sensors and Sensing Systems program, the National Science Foundation recently introduced a mechanism in which proposals are reviewed by other applicants and acceptance of an applicant's own proposal depends in part on the extent to which the *reviews* submitted by the applicant agree with other reviews of the same proposals. The specific mechanism used by the National Science Foundation was originally proposed by Merrifield and Saari [16] in the context of allocation of telescope time. Whether the mechanism provides the right incentives in peer review is debatable, but its lack of impartiality, which in this case is deliberate, would make it very hard to show any formal incentive properties. By contrast, our results allow for a separation of preferences regarding an agent's own selection and those regarding the selection of others, and can be combined in a straightforward way with peer prediction techniques [e.g., 18] to provide strict incentives for the truthful evaluation of other agents. The exact properties achievable by such hybrid mechanisms and their use in peer review deserve further investigation.

Impartial selection is also more distantly related to work in distributed computing on leader election [e.g., 1, 5, 9, 3] and work on the manipulation of reputation systems [e.g., 13]. Leader election seeks to guarantee the selection of a non-malicious agent in the presence of malicious agents trying to manipulate the selection process. Work on reputation systems often considers models with more complex preference and message spaces, where maximization of a one-dimensional objective does not suffice.

The 2-partition mechanism, finally, is reminiscent of random sampling in unlimited-supply auctions [11, 14, 10] and combinatorial auctions [8]. It will be interesting to see whether our more complicated mechanisms and analysis techniques can be applied to these settings in a meaningful way.

Open Problems. While we completely solve the general case and make significant progress for the special case without abstentions, several interesting directions for future work remain. The most obvious question of course concerns the gap for settings without abstentions between the lower bound of 67/108 provided by the permutation mechanism and the upper bound of 3/4. It is unknown whether the lower bound is tight, but a specialized upper bound of 2/3 for the permutation mechanism suggests that the latter may not be optimal. Alon et al. considered the more general problem of selecting any fixed number of agents and gave an α-optimal impartial mechanism where α tends to 1 as the number of agents to be selected tends to infinity. The question of optimal mechanisms for selecting a small number of agents is wide open. We may finally ask whether optimality and anonymity are incompatible. This question arises from the observation that the permutation mechanism considers agents one by one and thus cannot process nominations anonymously. The k-partition mechanism, on the other hand, allows nominations by agents from the same set to be considered simultaneously and thus offers a certain level of anonymity, but it is not optimal.

2. Preliminaries. For $n \in \mathbb{N}$, let

$$\mathcal{G}_n = \Big\{(N,E) : N = \{1,\dots,n\}, E \subseteq (N \times N) \setminus \bigcup\nolimits_{i \in N} (\{i\} \times \{i\})\Big\}$$

be the set of directed graphs with n vertices and no loops. Let $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$. For $G = (N,E) \in \mathcal{G}$, $S \subseteq N$, and $i \in N$, let $\delta^-_S(i,G) = |\{(j,i) \in E : G = (N,E), j \in S\}|$ denote the *indegree* of vertex i from vertices in S. We use $\delta^-(i,G)$ as a shorthand for $\delta^-_N(i,G)$, denote $\Delta(G) = \max_{i \in N} \delta^-(i,G)$, and write $\delta^-(i)$ instead of $\delta^-(i,G)$ and Δ instead of $\Delta(G)$ if G is clear from the context.

A *selection mechanism* for $\mathcal{G}$ is then given by a family of functions $f : \mathcal{G}_n \to [0,1]^n$ that maps each graph to a probability distribution on its vertices. In a slight abuse of notation, we use f to refer to both the mechanism and individual functions from the family. Mechanism f is *impartial* on $\mathcal{G}' \subseteq \mathcal{G}$ if on this set of graphs the probability of selecting vertex i does not depend on its outgoing edges, i.e., if for every pair of graphs $G = (N,E)$ and $G' = (N,E')$ in $\mathcal{G}'$ and every $i \in N$, $(f(G))_i = (f(G'))_i$ whenever $E \setminus (\{i\} \times N) = E' \setminus (\{i\} \times N)$. All mechanisms we consider are impartial on $\mathcal{G}$, and we simply refer to such mechanisms as impartial mechanisms. Mechanism f is α-*optimal* on $\mathcal{G}' \subseteq \mathcal{G}$, for $\alpha \le 1$, if for every graph in $\mathcal{G}'$ the expected indegree of the vertex selected by f differs from the maximum indegree by a factor of at most α, i.e., if

$$\inf_{\substack{G \in \mathcal{G} \\ \Delta(G) > 0}} \frac{\mathbb{E}_{i \sim f(G)}[\delta^-(i,G)]}{\Delta(G)} \ge \alpha.$$

We call a mechanism α-optimal if it is α-optimal on $\mathcal{G}$, and approximately optimal if it is α-optimal for some constant α.

As far as impartiality and approximate optimality are concerned, we can restrict our attention to symmetric mechanisms. Mechanism f is *symmetric* if it is invariant with respect to renaming of the vertices, i.e., if for every $G = (N,E) \in \mathcal{G}$, every $i \in N$, and every permutation $\pi = (\pi_1, \dots, \pi_{|N|})$ of N,

$$\big(f(G_\pi)\big)_{\pi_i} = \big(f(G)\big)_i,$$

where $G_\pi = (N, E_\pi)$ with $E_\pi = \{(\pi_i, \pi_j) : (i,j) \in E\}$. For a given mechanism f, denote by f_s the mechanism obtained by applying a random permutation π to the vertices of the input graph, invoking f, and permuting the result by the inverse of π, such that for all $n \in \mathbb{N}$, $G \in \mathcal{G}_n$, and $i \in \{1, \dots, n\}$,

$$\big(f_s(G)\big)_i = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \big(f(G_\pi)\big)_{\pi_i},$$

Input: Graph $G = (N, E)$
Output: Vertex $i \in N$

Assign each $i \in N$ independently and uniformly at random to one of two sets A_1 and A_2, such that $\mathbb{P}[i \in A_1] = \mathbb{P}[i \in A_2] = 1/2$ for all $i \in N$, $A_1 \cup A_2 = N$, and $A_1 \cap A_2 = \emptyset$;
if $A_2 = \emptyset$ **then** return a vertex chosen uniformly at random from N;
Return a vertex chosen uniformly at random from $\arg\max_{i \in A_2} \delta^-_{A_1}(i)$;

FIG. 1. *The 2-partition mechanism*

FIG. 2. *No impartial mechanism is more than 1/2-optimal*

where $\mathcal{S}_n$ is the set of all permutations $\pi = (\pi_1, \dots, \pi_n)$ of a set of n elements. The following result is straightforward.

LEMMA 2.1 (Holzman and Moulin [15]). *Let f be a selection mechanism that is impartial and α-optimal on $\mathcal{G}' \subseteq \mathcal{G}$. Then f_s is impartial, α-optimal, and symmetric on $\mathcal{G}'$.*

3. The 2-Partition Mechanism. We begin our investigation with a more detailed analysis of the 2-partition mechanism proposed by Alon et al. [2]. The mechanism first assigns each vertex independently and uniformly at random to one of two sets A_1 and A_2. Then it returns a vertex from A_2 that has maximum indegree from vertices in A_1, or a vertex chosen uniformly at random from N in case $A_2 = \emptyset$. A formal description of the mechanism is given in Figure 1.

The 2-partition mechanism is obviously impartial, as the outgoing edges of vertex $i \in N$ can influence the outcome only if $i \in A_1$ and $A_2 \neq \emptyset$, in which case i will never be selected. It is also easy to see that the mechanism is 1/4-optimal. As noted by Alon et al., for an arbitrary graph G and a particular vertex i^* of G with indegree $\Delta = \Delta(G)$, we have that $\mathbb{P}[i^* \in A_2] = 1/2$ and, by linearity of expectation, $\mathbb{E}[\delta^-_{A_1}(i^*) \mid i^* \in A_2] = \mathbb{E}[\delta^-_{A_1}(i^*)] = \delta^-_N(i^*)/2 = \Delta/2$. The expected indegree of the selected vertex is thus at least $\Delta/2$ with probability at least 1/2, i.e., at least $\Delta/4$. A graph with a single edge shows that this result is in fact tight. Alon et al. noted further that no impartial mechanism can be more than 1/2-optimal. To this end, consider the two graphs in Figure 2, and the probabilities p_1, p_2, and p_3 with which certain vertices in these graphs are selected. Due to symmetry, which we can assume by Lemma 2.1, $p_1 = p_2$ and thus $p_1 \leq 1/2$. On the other hand, $p_1 = p_3$ by impartiality, so the expected indegree of the vertex selected in the right graph is at most 1/2 and the claim follows.

The rather straightforward analysis of the 2-partition mechanism does not lead to a tight result, but it is unsatisfactory in particular because it provides no information about the performance of the mechanism on more complicated graphs, and no cues what a better mechanism might look like. We will gain both from the proof of the following lemma, which establishes a lower bound on the expected indegree of the selected vertex relative to the maximum indegree $\Delta(G)$.

LEMMA 3.1. *On any graph G with maximum indegree $\Delta = \Delta(G)$, the 2-partition mechanism is $\alpha_2(\Delta)$-optimal, where $\alpha_2(\Delta) = \frac{1}{\Delta 2^\Delta} \sum_{k=0}^{\Delta} \binom{\Delta}{k} \min\{\frac{\Delta}{2}, k\}$.*

Proof. Let $i^* \in N$ such that $\delta^-(i^*) = \Delta(G)$, and denote by X the indegree of the vertex selected by the 2-partition mechanism. Then X is a random variable subject to the internal randomness of the mechanism, and we will be interested in its expected value $\mathbb{E}[X]$.

Let $\boldsymbol{A} = (A_1, A_2)$ be the partition selected in Line 1 of the 2-partition mechanism in Figure 1, and consider an arbitrary set $S \subseteq N \setminus \{i^*\}$ of vertices other than i^*. We begin by bounding $\mathbb{E}[X \mid A_1 \setminus \{i^*\} = S]$, i.e., the expected value of X given that $A_1 = S$ or $A_1 = S \cup \{i^*\}$. To this end, let $v(S)$ and $a(S)$ respectively denote the indegree of i^* from S and the maximum indegree of any *other* vertex in $N \setminus S$ from S, i.e., $v(S) = \delta^-_S(i^*)$ and $a(S) = \max_{i \in N \setminus (S \cup \{i^*\})} \delta^-_S(i)$.

Assume for now that $S \neq \emptyset$ and $S \neq N \setminus \{i^*\}$. Then, $\mathbb{E}[X \mid A_1 = S] = \Delta$ if $v(S) > a(S)$, $\mathbb{E}[X \mid A_1 = S] \geq a(S)$ if $a(S) \geq v(S)$, and $\mathbb{E}[X \mid A_1 = S \cup \{i^*\}] \geq a(S)$. To see this, recall that X is the indegree of the selected vertex from vertices in N, and note that the expected value of X only increases if there is an edge from i^* to a vertex for which $a(S)$ is attained. Since the events where $A_1 = S$ and $A_1 = S \cup \{i^*\}$ occur with equal probability,

$$\begin{aligned}\mathbb{E}[X \mid A_1 \setminus \{i^*\} = S] &\geq \frac{\chi\big[v(S) > a(S)\big]\,\Delta + \big(1 - \chi\big[v(S) > a(S)\big]\big)\,a(S)}{2} + \frac{a(S)}{2} \\ &= a(S) + \frac{1}{2}\,\chi\big[v(S) > a(S)\big]\big(\Delta - a(S)\big),\end{aligned}$$

where χ denotes the indicator function on binary events, i.e., $\chi[E] = 1$ if event E takes place and $\chi[E] = 0$ otherwise. Given any fixed value of $v(S)$, the right-hand side of this expression is a linearly increasing function of $a(S)$ except for a possible discontinuity at $a(S) = v(S)$. It is thus minimized either at the leftmost point of its domain, where $a(S) = 0$, or at the point of discontinuity, where $a(S) = v(S)$. Its value is $\chi[v(S) > 0] \cdot \Delta/2$ in the former case and $v(S)$ in the latter, so in summary

$$\mathbb{E}[X \mid A_1 \setminus \{i^*\} = S] \geq \min\Big\{\chi\big[v(S) > 0\big] \cdot \frac{\Delta}{2}, v(S)\Big\} = \min\Big\{\frac{\Delta}{2}, v(S)\Big\}. \tag{3.1}$$

We can now lift the assumption that $S \neq \emptyset$ and $S \neq N \setminus \{i^*\}$. If $S = \emptyset$, then $v(S) = 0$ and (3.1) holds trivially. If $S = N \setminus \{i^*\}$, then $v(S) = \Delta$, and i^* is in $N \setminus S$ and therefore chosen by the 2-partition mechanism with probability $1/2$. Thus $\mathbb{E}[X \mid A_1 \setminus \{i^*\} = S] \geq \Delta/2 = \min\{\Delta/2, v(S)\}$, and (3.1) is again satisfied.

By construction of the 2-partition mechanism, each vertex belongs to A_1 with probability $1/2$, so $v(S) = \delta^-_{A_1}(i^*)$ is distributed according to the binomial distribution with Δ trials and success probability $1/2$. We thus have that

$$\begin{aligned}\mathbb{E}[X] &= \sum_{S \subseteq N} \mathbb{P}[A_1 \setminus \{i^*\} = S] \cdot \mathbb{E}[X \mid A_1 \setminus \{i^*\} = S] \\ &\geq \sum_{k=0}^{\Delta} \sum_{\substack{S \subseteq N \\ v(S) = k}} \mathbb{P}[A_1 \setminus \{i^*\} = S] \cdot \min\Big\{\frac{\Delta}{2}, k\Big\} \\ &= \frac{1}{2^\Delta} \sum_{k=0}^{\Delta} \binom{\Delta}{k} \min\Big\{\frac{\Delta}{2}, k\Big\}.\end{aligned}$$

This finally implies that $\alpha_2(\Delta) \geq \frac{1}{\Delta 2^\Delta} \sum_{k=0}^{\Delta} \binom{\Delta}{k} \min\{\frac{\Delta}{2}, k\}$ as claimed. $\square$

Using Lemma 3.1 it is straightforward if somewhat tedious to derive the following closed-form expression for $\alpha_2(\Delta)$, a rigorous proof can be found in Appendix A.

THEOREM 3.2. *On any graph G with maximum indegree $\Delta = \Delta(G)$, the 2-partition mechanism is $\alpha_2(\Delta)$-optimal, where*

$$\alpha_2(\Delta) = \begin{cases} \frac{1}{4} & \text{if } \Delta = 1, \\ \frac{1}{2} - \frac{1}{2^{\Delta+2}}\binom{\Delta}{\Delta/2} & \text{if } \Delta \geq 2 \text{ and even}, \\ \alpha_2(\Delta - 1) & \text{if } \Delta \geq 3 \text{ and odd.} \end{cases}$$

Given its closed form, it is easy to show that $\alpha_2(\Delta)$ is non-decreasing in Δ.

COROLLARY 3.3. *For every $\Delta \in \mathbb{N}$, $\alpha_2(\Delta+1) \geq \alpha_2(\Delta)$ and $\alpha_2(\Delta+2) > \alpha_2(\Delta)$.*

Proof. Since $\alpha_2(\Delta) = \alpha_2(\Delta - 1)$ for odd $\Delta \geq 3$ by Theorem 3.2, it suffices to show that $\alpha_2(\Delta) > \alpha_2(\Delta - 2)$ for even $\Delta \geq 4$. To see this, note that $\alpha_2(\Delta) = \frac{1}{2} - \frac{1}{2^{\Delta+2}}\binom{\Delta}{\Delta/2}$. Using three times that $\binom{\Delta}{k} = \binom{\Delta-1}{k-1} + \binom{\Delta-1}{k}$, we obtain

$$\alpha_2(\Delta) = \frac{1}{2} - \frac{1}{2^{\Delta+2}}\left(\binom{\Delta-2}{\Delta/2-2} + 2\binom{\Delta-2}{\Delta/2-1} + \binom{\Delta-2}{\Delta/2}\right),$$

and since $\binom{\Delta-2}{k}$ is maximized for $k = \Delta/2 - 1$,

$$\alpha_2(\Delta) > \frac{1}{2} - \frac{1}{2^{\Delta}}\binom{\Delta-2}{\Delta/2-1} = \alpha_2(\Delta - 2). \qquad \square$$

These results imply that a graph with a single edge is in fact the unique worst case for the 2-partition mechanism, and they also yield the first non-trivial lower bound for settings without abstentions. In the absence of abstentions, one of two conditions is always satisfied: either every vertex has indegree exactly one, in which case every mechanism including the 2-partition mechanism is optimal, or $\Delta \geq 2$ and the 2-partition mechanism is $\alpha_2(2)$-optimal. Since $\alpha_2(2) = 3/8$, we conclude that the 2-partition mechanism is 3/8-optimal on all instances without abstentions. We will return to this special case, and show a better bound, in Section 7.

4. The k-Partition Mechanism. What is perhaps most interesting about the above analysis of the 2-partition mechanism is that the same technique can in principle also be applied to a partition of the vertices into more than two sets. Indeed, in this section, we propose a generalization of the 2-partition mechanism to a larger number of sets and then generalize the analysis technique to the new mechanism.

For a fixed $k \geq 2$, the new mechanism first assigns each vertex $i \in N$ independently and uniformly at random to one of k sets $A_1, \dots, A_k$. The mechanism then proceeds in k iterations, during which it maintains and updates a candidate vertex that is finally selected after iteration k. In the jth iteration, the candidate is updated if the maximum indegree among vertices in A_j from vertices in $A_{<j} = \bigcup_{i=1}^{j-1} A_i$ *other than the candidate* is at least that of the candidate at the time it became the candidate. In that case, the new candidate is chosen uniformly at random from the set of vertices in A_j with maximum indegree from vertices in $A_{<j} = \bigcup_{i=1}^{j-1} A_i$, *now including the previous candidate*. The mechanism is clearly impartial, because it only takes into account the outgoing edges of vertices that can no longer be selected. That the outgoing edges of the previous candidate are taken into account when selecting the new candidate is somewhat subtle, but it turns out to be crucial. A formal description of the mechanism is given in Figure 3.

Input: Graph $G = (N, E)$
Output: Vertex $i \in N$

Assign each $i \in N$ independently and uniformly at random to one of k sets $A_1, \dots, A_k$, such that $\mathbb{P}[i \in A_j] = 1/k$ for all $i \in N$ and $j \in \{1, \dots, k\}$, $\bigcup_{j=1}^{k} A_j = N$, and $A_j \cap A_\ell = \emptyset$ for all $j, \ell \in \{1, \dots, k\}$ with $j \neq \ell$;
Set $\{i^*\} := \emptyset$, $d^* := 0$;
for $j = 1, \dots, k$ **do**
  if $\max_{i \in A_j} \delta^-_{A_{<j} \setminus \{i^*\}}(i) \geq d^*$ **then**
    Choose $i^* \in \arg\max_{i \in A_j} \delta^-_{A_{<j}}(i)$ uniformly at random;
    set $d^* := \delta^-_{A_{<j}}(i^*)$;
Return i^*;

Fig. 3. *The k-partition mechanism*

Now consider a graph $G = (N, E) \in \mathcal{G}$ and a vertex $i^* \in N$ with indegree $\Delta = \Delta(G)$. Fix $k \in \mathbb{N}$, and let X be the indegree of the vertex selected from G by the k-partition mechanism. Note that X is a random variable subject to the internal randomness of the mechanism, and that we are interested in bounding its expected value from below. We begin by bounding its conditional expectation given that a partition of all vertices except i^* is fixed and i^* is assigned uniformly at random to one of the sets of the partition.

We need some notation. For $N' \subseteq N$, let $\mathcal{P}_k(N')$ denote the set of all partitions $\boldsymbol{S} = (S_1, \dots, S_k)$ of N' into k possibly empty sets $S_1, \dots, S_k$. For a partition $\boldsymbol{S} = (S_1, \dots, S_k)$ and $j \in \{1, \dots, k\}$, let $S_{<j} = \bigcup_{\ell=1}^{j-1} S_\ell$. For $\boldsymbol{S} \in \mathcal{P}_k(N)$ and $i \in N$, we slightly abuse notation and write $\boldsymbol{S} \setminus \{i\} = (S_1 \setminus \{i\}, \dots, S_k \setminus \{i\})$ for the partition of $N \setminus \{i\}$ obtained by removing i from the set in $\boldsymbol{S}$ it is a member of. The bounds we are about to derive can be expressed compactly as a minimum over the entries of a vector, which can in turn be written as the product $\boldsymbol{W}^k \cdot \boldsymbol{v}(\boldsymbol{S})$ between a certain matrix $\boldsymbol{W}^k$ and a vector $\boldsymbol{v}(\boldsymbol{S})$ with entry $v_j(\boldsymbol{S})$ equal to the indegree of i^* from $\boldsymbol{S}_j$. For this, define $\boldsymbol{W}^k$ as the $k \times k$ matrix with entries in row j on and above the diagonal equal to $\frac{k-j}{k}$, and entries below the diagonal equal to 1, i.e.,

$$\boldsymbol{W}^k = (w^k_{ij})_{i,j=1,\dots,k} = \begin{pmatrix} \frac{k-1}{k} & \frac{k-1}{k} & \frac{k-1}{k} & \dots & \frac{k-1}{k} \\ 1 & \frac{k-2}{k} & \frac{k-2}{k} & \dots & \frac{k-2}{k} \\ 1 & 1 & \frac{k-3}{k} & \dots & \frac{k-3}{k} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

We obtain the following result.

Lemma 4.1. *Consider a graph $G = (N, E)$ and a vertex i^* with indegree $\Delta = \Delta(G)$. Let X be the indegree of the vertex selected by the k-partition mechanism from G, let $\boldsymbol{S} = (S_1, \dots, S_k) \in \mathcal{P}_k(N \setminus \{i^*\})$, and $\boldsymbol{v}(\boldsymbol{S}) = (v_j(\boldsymbol{S}))_{j=1,\dots,k}$ with $v_j(\boldsymbol{S}) = \delta^-_{S_j}(i^*)$. Then, $\mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] \geq \min\{\boldsymbol{W}^k \cdot \boldsymbol{v}(\boldsymbol{S})\}$.*

Proof. Fix a partition $\boldsymbol{S} \in \mathcal{P}_k(N \setminus \{i^*\})$ and denote by

$$a(\boldsymbol{S}) = \max_{j=1,\dots,k} \max_{i \in S_j} \delta^-_{S_{<j}}(i) \tag{4.1}$$

the maximum indegree of any vertex, with the exception of i^*, from vertices in sets with smaller index than its own. For $j \in \{1, \dots, k\}$, let $v_{<j}(\boldsymbol{S}) = \sum_{\ell=1}^{j-1} v_\ell(\boldsymbol{S})$ denote the indegree of i^* from vertices in the first $j-1$ sets. We claim that the k-partition mechanism (i) always selects a vertex with indegree at least $a(\boldsymbol{S})$, and (ii) selects vertex i^* if $i^* \in A_j$ and $v_{<j}(\boldsymbol{S}) > a(\boldsymbol{S})$ for some $j \in \{1, \dots, k\}$, such that

$$\mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}, i^* \in A_j] \geq a(\boldsymbol{S}) + \chi\big[v_{<j}(\boldsymbol{S}) > a(\boldsymbol{S})\big]\big(\Delta - a(\boldsymbol{S})\big). \tag{4.2}$$

For the first part of the claim, fix a vertex i' and an index j' for which the maximum in (4.1) is attained. In iteration j' the mechanism then considers at least $a(\boldsymbol{S}) - 1$ incoming edges for vertex i', namely those from vertices in $\bigcup_{\ell=1}^{j'-1} A_\ell$ with the exception of the current candidate. The mechanism then chooses a new candidate unless the current candidate itself has indegree at least $a(\boldsymbol{S})$. In case a new candidate is chosen, the outgoing edges of the previous candidate are taken into account, so that at least $a(\boldsymbol{S})$ incoming edges are considered for vertex i'. By the same reasoning, a new candidate can only be chosen in later iterations if it has indegree at least $a(\boldsymbol{S})$, which eventually leads to the selection of a vertex with at least that indegree.

For the second part of the claim assume that $i^* \in A_j$ and $v_{<j}(\boldsymbol{S}) > a(\boldsymbol{S})$ for some $j \in \{1, \dots, k\}$. It is then easy to see that i^* is chosen in iteration j of the mechanism and remains the candidate until the mechanism terminates.

From (4.2) and the fact that $\mathbb{P}[i^* \in A_j] = 1/k$ for $j = 1, \dots, k$ we obtain that

$$\begin{aligned}\mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] &= \frac{1}{k} \sum_{j=1}^{k} \mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}, i^* \in A_j] \\ &\geq a(\boldsymbol{S}) + \frac{\Delta - a(\boldsymbol{S})}{k} \sum_{m=1}^{k} \chi\big[v_{<m}(\boldsymbol{S}) > a(\boldsymbol{S})\big].\end{aligned}$$

Given fixed values $v_{<m}(\boldsymbol{S})$ for $m = 1, \dots, k$, the right-hand side of this expression is a linearly increasing function of $a(\boldsymbol{S})$ except for possible discontinuities at $v_{<2}(\boldsymbol{S}), \dots, v_{<m}(\boldsymbol{S})$. It is thus minimized either at 0, or when $a(\boldsymbol{S}) = v_{<j}(\boldsymbol{S})$ for some $j \in \{2, \dots, k\}$, so

$$\begin{aligned}\mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] &\geq \min_{j=1,\dots,k} \left\{ v_{<j}(\boldsymbol{S}) + \frac{\Delta - v_{<j}(\boldsymbol{S})}{k} \sum_{m=1}^{k} \chi\big[v_{<m}(\boldsymbol{S}) > v_{<j}(\boldsymbol{S})\big] \right\} \\ &= \min_{j=1,\dots,k} \left\{ v_{<j}(\boldsymbol{S}) + \frac{\Delta - v_{<j}(\boldsymbol{S})}{k} \sum_{m=j+1}^{k} \chi\big[v_{<m}(\boldsymbol{S}) > v_{<j}(\boldsymbol{S})\big] \right\}.\end{aligned}$$

Now observe that for any $\ell \in \{1, \dots, k-1\}$, the terms for $j = \ell$ and $j = \ell + 1$ are equal when $v_{<\ell}(\boldsymbol{S}) = v_{<\ell+1}(\boldsymbol{S})$. The minimum will thus be obtained for $j = k$, or for some $j \in \{1, \dots, k-1\}$ with $v_{<j}(\boldsymbol{S}) < v_{<j+1}(\boldsymbol{S})$. Adopting the convention that $v_{<k+1}(\boldsymbol{S}) = \infty$, we may express this compactly as

$$\begin{aligned}\mathbb{E}[X \mid \boldsymbol{A} \setminus \{i^*\} = \boldsymbol{S}] &\geq \min_{\substack{j=1,\dots,k \\ v_{<j}(\boldsymbol{S}) < \\ v_{<j+1}(\boldsymbol{S})}} \left\{ v_{<j}(\boldsymbol{S}) + \frac{\Delta - v_{<j}(\boldsymbol{S})}{k} \sum_{m=j+1}^{k} \chi\big[v_{<m}(\boldsymbol{S}) > v_{<j}(\boldsymbol{S})\big] \right\} \\ &= \min_{\substack{j=1,\dots,k \\ v_{<j}(\boldsymbol{S}) < \\ v_{<j+1}(\boldsymbol{S})}} \left\{ v_{<j}(\boldsymbol{S}) + \frac{k-j}{k}(\Delta - v_{<j}(\boldsymbol{S})) \right\}.\end{aligned}$$

Since $v_{<j}(\boldsymbol{S})+\frac{k-j}{k}(\Delta-v_{<j}(\boldsymbol{S}))$ never attains its minimum for $j\in\{1,\dots,k-1\}$ with $v_{<j}(\boldsymbol{S})=v_{<j+1}(\boldsymbol{S})$, we may drop the condition that $v_{<j}(\boldsymbol{S})<v_{<j+1}(\boldsymbol{S})$ and obtain

$$\mathbb{E}[X \mid \boldsymbol{A}\setminus\{i^*\}=\boldsymbol{S}] \geq \min_{j=1,\dots,k}\left\{v_{<j}(\boldsymbol{S})+\frac{k-j}{k}(\Delta-v_{<j}(\boldsymbol{S}))\right\}.$$

Since $\Delta=\sum_{\ell=1}^{k} v_\ell(\boldsymbol{S})$,

$$\mathbb{E}[X \mid \boldsymbol{A}\setminus\{i^*\}=\boldsymbol{S}] \geq \min_{j=1,\dots,k}\left\{\sum_{\ell=1}^{j-1} v_\ell(\boldsymbol{S})+\sum_{\ell=j}^{k}\frac{k-j}{k}v_\ell(\boldsymbol{S})\right\}=\min\{\boldsymbol{W}^k\cdot\boldsymbol{v}(\boldsymbol{S})\}$$

as claimed. ∎

To obtain a bound on $\mathbb{E}[X]$, we now average the expression in Lemma 4.1 over the distribution on partitions of N. For $\Delta,k\in\mathbb{N}$, let $P_k(\Delta)=\{\boldsymbol{v}\in\mathbb{N}^k:\sum_{j=1}^{k}v_j=\Delta\}$. For $\boldsymbol{v}\in P_k(\Delta)$, let $\binom{\Delta}{\boldsymbol{v}}=\frac{\Delta!}{v_1!\cdots v_k!}$ denote the number of partitions of a set with Δ elements into k sets of sizes $v_1,\dots,v_k$. The following is a straightforward corollary of Lemma 4.1.

Lemma 4.2. *On any graph G with maximum indegree $\Delta=\Delta(G)$, the k-partition mechanism is $\alpha_k(\Delta)$-optimal, where* $\alpha_k(\Delta)=\frac{1}{\Delta k^\Delta}\sum_{\boldsymbol{v}\in P_k(\Delta)}\binom{\Delta}{\boldsymbol{v}}\min\{\boldsymbol{W}^k\cdot\boldsymbol{v}\}$.

Proof. Consider a vertex i^* with indegree Δ. Then

$$\begin{aligned}\mathbb{E}[X]&=\sum\nolimits_{\boldsymbol{S}\in\mathcal{P}_k(N\setminus\{i^*\})}\mathbb{P}[\boldsymbol{A}\setminus\{i^*\}=\boldsymbol{S}]\cdot\mathbb{E}[X\mid\boldsymbol{A}\setminus\{i^*\}=\boldsymbol{S}],\\&=\sum\nolimits_{\boldsymbol{v}\in P_k(\Delta)}\sum\nolimits_{\substack{\boldsymbol{S}\in\mathcal{P}_k(N\setminus\{i^*\})\\ \delta^-_{S_i}(i^*)=v_i,\ i=1,\dots,k}}\mathbb{P}[\boldsymbol{A}\setminus\{i^*\}=\boldsymbol{S}]\cdot\mathbb{E}[X\mid\boldsymbol{A}\setminus\{i^*\}=\boldsymbol{S}]\\&\geq\sum\nolimits_{\boldsymbol{v}\in P_k(\Delta)}\frac{1}{k^\Delta}\binom{\Delta}{\boldsymbol{v}}\min\{\boldsymbol{W}^k\cdot\boldsymbol{v}\},\end{aligned}$$

where the inequality holds by Lemma 4.1. ∎

In analyzing the 2-partition mechanism, we derived a closed-form expression for $\alpha_2(\Delta)$ that turned out to be monotonically non-decreasing in Δ. While the complexity of α_k prevents us from taking the same route for $k>2$, monotonicity turns out to hold for any value of k.

Lemma 4.3. *For any $k\geq 2$, $\alpha_k(\Delta)$ is non-decreasing in Δ.*

Proof. Denoting the the jth row of $\boldsymbol{W}^k$ by $\boldsymbol{W}^k_j$, we can reformulate the lower bound of Lemma 4.2 as

$$\alpha_k(\Delta)=\frac{1}{\Delta k^\Delta}\sum\nolimits_{\boldsymbol{v}\in P_k(\Delta)}\binom{\Delta}{\boldsymbol{v}}\min_{j=1,\dots,k}\langle\boldsymbol{W}^k_j,\boldsymbol{v}\rangle.$$

Instead of summing over all vectors $\boldsymbol{v}\in P_k(\Delta)$, we may instead sum over all vectors $\boldsymbol{v}\in P_k(\Delta+1)$ and decrease one of the non-zero entries of $\boldsymbol{v}$ by 1. Thus

$$\alpha_k(\Delta)=\frac{1}{\Delta k^{\Delta+1}}\sum\nolimits_{\boldsymbol{v}\in P_k(\Delta+1)}\frac{\binom{\Delta+1}{\boldsymbol{v}}}{\Delta+1}\sum_{i=1}^{k}v_i\min_{j=1,\dots,k}\langle\boldsymbol{W}^k_j,\boldsymbol{v}-\boldsymbol{e}^{k,i}\rangle,$$

where $\boldsymbol{e}^{k,i}$ is the ith unit vector in k dimensions, i.e., $e^{k,i}_\ell=1$ if $\ell=i$ and $e^{k,i}_\ell=0$, otherwise. If we exchange the order of the summation over i and the minimization over j, the value of the expression can only increase, so

$$\alpha_k(\Delta)\leq\frac{1}{\Delta k^{\Delta+1}}\sum\nolimits_{\boldsymbol{v}\in P_k(\Delta+1)}\frac{\binom{\Delta+1}{\boldsymbol{v}}}{\Delta+1}\min_{j=1,\dots,k}\sum_{i=1}^{k}v_i\,\langle\boldsymbol{W}^k_j,\boldsymbol{v}-\boldsymbol{e}^{k,i}\rangle$$

$$= \frac{1}{(\Delta+1)k^{\Delta+1}} \sum\nolimits_{\boldsymbol{v} \in P_k(\Delta+1)} \frac{\binom{\Delta+1}{\boldsymbol{v}}}{\Delta} \min_{j=1,\dots,k} \sum_{i=1}^{k} \Big(v_i \langle \boldsymbol{W}_j^k, \boldsymbol{v} \rangle - v_i \left\langle \boldsymbol{W}_j^k, \boldsymbol{e}^{k,i} \right\rangle \Big)$$

$$= \frac{1}{(\Delta+1)k^{\Delta+1}} \sum\nolimits_{\boldsymbol{v} \in P_k(\Delta+1)} \frac{\binom{\Delta+1}{\boldsymbol{v}}}{\Delta} \min_{j=1,\dots,k} \left(\langle \boldsymbol{W}_j^k, \boldsymbol{v} \rangle \sum_{i=1}^{k} v_i - \langle \boldsymbol{W}_j^k, \boldsymbol{v} \rangle \right)$$

$$= \frac{1}{(\Delta+1)k^{\Delta+1}} \sum\nolimits_{\boldsymbol{v} \in P_k(\Delta+1)} \binom{\Delta+1}{\boldsymbol{v}} \min_{j=1,\dots,k} \langle \boldsymbol{W}_j^k, \boldsymbol{v} \rangle$$

$$= \alpha_k(\Delta+1). \qquad \square$$

Monotonicity of α_k allows us to obtain a lower bound on the approximation ratio of the k-partition mechanism by bounding $\alpha_k(1)$ from below.

THEOREM 4.4. *The k-partition mechanism for $k \geq 2$ is $\frac{k-1}{2k}$-optimal.*

Proof. In light of Lemma 4.3, it suffices to show that $\alpha_k(1) \geq \frac{k-1}{2k}$ for every $k \geq 2$. By Lemma 4.2,

$$\alpha_k(1) = \frac{1}{k} \sum\nolimits_{\boldsymbol{v} \in P_k(1)} \min\{\boldsymbol{W}^k \cdot \boldsymbol{v}\}.$$

Taking the pointwise minimum for each row of $\boldsymbol{W}^k$,

$$\alpha_1(k) \geq \frac{1}{k} \sum\nolimits_{\boldsymbol{v} \in P_k(1)} \left\langle \boldsymbol{v}, \left(\frac{k-1}{k}, \frac{k-2}{k}, \dots, \frac{1}{k}, 0 \right) \right\rangle.$$

In the sum every unit vector occurs exactly once, and thus

$$\alpha_k(1) \geq \frac{1}{k} \sum_{i=1}^{k} \frac{k-i}{k} = \frac{1}{k^2} \sum_{i=0}^{k-1} i = \frac{k(k-1)}{2k^2} = \frac{k-1}{2k}. \qquad \square$$

5. The Permutation Mechanism. We have started from the simple result that no impartial selection mechanism can be more than 1/2-optimal, and in the previous section identified a class of mechanisms parameterized by $k \in \mathbb{N}$ that attains this bound in the limit as k tends to infinity. It turns out that the bound can also be attained exactly, by a limiting mechanism for the above class. This mechanism, which we call the permutation mechanism, considers the vertices one by one according to a random permutation $\pi = (\pi_1, \dots, \pi_n)$ and in each step compares the current vertex π_j to a single candidate vertex π_ℓ with $\ell < j$. In determining the indegree of the candidate vertex π_ℓ it takes into account the outgoing edges of vertices $\pi_1, \dots, \pi_{\ell-1}$. For the indegree of the current vertex π_j it takes into account the outgoing edges of vertices $\pi_1, \dots, \pi_{j-1}$, except π_ℓ. If the latter is greater than or equal than the former, π_j becomes the new candidate vertex, and the candidate vertex after the final step is the one selected by the mechanism. A formal description of the mechanism is given in Figure 4.

It is again easy to see that the mechanism is impartial, and we obtain the following performance guarantee.

THEOREM 5.1. *The permutation mechanism is* 1/2*-optimal.*

Proof. Assume for contradiction that there exists a graph $G = (N, E)$ such that the permutation mechanism is strictly less than 1/2-optimal on G. Let $n = |N|$ and $\Delta = \Delta(G)$, and denote by X and by X_k for $k \geq 2$ the indegrees of the vertices respectively selected from G by the permutation and the k-partition mechanism. Note that

Input: Graph $G = (N, E)$
Output: Vertex $i \in N$
Choose a permutation $(\pi_1, \ldots, \pi_{|N|})$ of N uniformly at random; for $j \in \{1, \ldots, |N|\}$ denote $\pi_{<j} = \{\pi_1, \ldots, \pi_{j-1}\}$;
Set $i^* := \pi_1$, $d^* := 0$;
for $j = 2, \ldots, |N|$ **do**
  if $\delta^-_{\pi_{<j} \setminus \{i^*\}}(\pi_j) \geq d^*$ **then**
    Set $i^* := \pi_j$, $d^* := \delta^-_{\pi_{<j}}(\pi_j)$;
return i^*;

FIG. 4. *The permutation mechanism*

X and X_k are random variables subject to the internal randomness of the respective mechanism. Finally let $\alpha = \mathbb{E}[X]/\Delta$, and note that $\alpha < 1/2$ by assumption.

For any k, the outcomes of the permutation mechanism and the k-partition mechanism agree under the condition that the partition $(A_1, \ldots, A_k)$ chosen by the latter satisfies $|A_i| \leq 1$ for $i = 1, \ldots, k$, so

$$\mathbb{E}[X] \geq \mathbb{P}\big[|A_i| \leq 1 \text{ for all } i \in \{1 \ldots, k\}\big] \cdot \mathbb{E}[X_k].$$

For any $k \geq n$,

$$\mathbb{P}\big[|A_i| \leq 1 \text{ for all } i \in \{1 \ldots, k\}\big] = \frac{k \cdot (k-1) \cdot \ \ldots \ \cdot (k-n+1)}{k^n} \geq \frac{(k-n)^n}{k^n},$$

and thus

$$\mathbb{E}[X] \geq \frac{(k-n)^n}{k^n}\, \mathbb{E}[X_k] \geq \frac{(k-n)^n}{k^n} \cdot \frac{k-1}{2k}\, \Delta,$$

where the second inequality follows from Theorem 4.4. For any fixed n,

$$\lim_{k \to \infty} \left(\frac{(k-n)^n}{k^n} \cdot \frac{k-1}{2k} \right) = \frac{1}{2} > \alpha,$$

and we can choose k such that

$$\frac{(k-n)^n}{k^n} \cdot \frac{k-1}{2k} > \alpha.$$

Thus $\mathbb{E}[X] > \alpha\Delta$, a contradiction. ∎

A potential downside of the permutation mechanism is that it considers agents one by one and therefore cannot process nominations anonymously. This may be of concern in situations where agents do not want their opinion regarding other agents to be publicly known. In the k-partition mechanism for some fixed value of k, on the other hand, the nominations submitted by agents in block A_j of the partition can be processed simultaneously and thus with partial anonymity. It is an interesting question whether this tradeoff between anonymity and approximate optimality is intrinsic to the problem, or whether there exits a different mechanism that provides similar performance guarantees as the permutation mechanism but a greater level of anonymity.

6. Highly Nominated Agents. Bousquet et al. [4] recently studied the asymptotic performance of impartial mechanisms and obtained the following result, among others: for any $\epsilon > 0$, there exists $\Delta_\epsilon \in \mathbb{N}$ such that the permutation mechanism is $(3/4-\epsilon)$-optimal on graphs with maximum indegree at least Δ_ϵ. For the sake of completeness, we show how this result can be obtained from Lemma 4.2 by a straightforward application of Chebychev's inequality. We state the result more generally for the k-partition mechanism, the analogous result for the permutation mechanism then follows from the same argument as in Section 5.

Theorem 6.1. *Let $\epsilon > 0$, $k \in \mathbb{N}$ with $k \geq 2$, and consider any graph G with maximum indegree $\Delta = \Delta(G) \geq 8k^2/\epsilon^3$. Then the k-partition mechanism is $(1-\epsilon)(\frac{3}{4} - \frac{1}{2k} - \frac{k \bmod 2}{4k^2})$-optimal on G.*

Proof. Recall that by Lemma 4.2,

$$\alpha_k(\Delta) = \frac{1}{\Delta k^\Delta} \sum_{\boldsymbol{v}\in P_k(\Delta)} \binom{\Delta}{\boldsymbol{v}} \min\{\boldsymbol{W}^k \cdot \boldsymbol{v}\}.$$

Given $\boldsymbol{v} \in P_k(\Delta)$ chosen uniformly at random, $\mathbb{E}[v_i] = \Delta/k$ for each $i \in \{1,\dots,k\}$. We will restrict attention to the case where $v_i > \frac{\Delta}{k}(1-\frac{\epsilon}{2})$ for all $i \in \{1,\dots,k\}$, and exploit that this happens with high probability for any fixed $\epsilon > 0$ when Δ is sufficiently large. Let $\epsilon > 0$ and observe that

$$\begin{aligned}\alpha_k(\Delta) \geq \frac{1}{\Delta k^\Delta} \sum_{\boldsymbol{v}\in P_k(\Delta)} \chi\Big[v_i > \frac{\Delta}{k}\Big(1-\frac{\epsilon}{2}\Big) \text{ for all } i \in \{1,\dots,k\}\Big] \\ \cdot \binom{\Delta}{\boldsymbol{v}} \frac{\Delta}{k}\Big(1-\frac{\epsilon}{2}\Big) \min\{\boldsymbol{W}^k \cdot \mathbf{1}^k\},\end{aligned}$$

where $\mathbf{1}^k = (1,\dots,1)$ is the all-ones vector in k dimensions.

It is easily verified that $\min\{\boldsymbol{W}^k \cdot \mathbf{1}^k\} = \langle \boldsymbol{W}^k_{\lceil k/2\rceil}, \mathbf{1}^k\rangle$ and

$$\langle \boldsymbol{W}^k_{\lceil k/2\rceil}, \mathbf{1}^k\rangle = \begin{cases} \frac{3}{4}k - \frac{1}{2} & \text{if } k \text{ is even} \\ \frac{3}{4}k - \frac{1}{2} - \frac{1}{4k} & \text{otherwise.}\end{cases}$$

Thus

$$\begin{aligned}\alpha_k(\Delta) &\geq \frac{1}{\Delta k^\Delta}\Big(\frac{3}{4}k - \frac{1}{2} - \frac{k \bmod 2}{4k}\Big)\frac{\Delta}{k}\Big(1-\frac{\epsilon}{2}\Big) \\ &\qquad \cdot \sum_{\boldsymbol{v}\in P_k(\Delta)} \chi\Big[v_i > \frac{\Delta}{k}\Big(1-\frac{\epsilon}{2}\Big) \text{ for all } i \in \{1,\dots,k\}\Big]\binom{\Delta}{\boldsymbol{v}} \\ &= \Big(1-\frac{\epsilon}{2}\Big)\Big(\frac{3}{4} - \frac{1}{2k} - \frac{k \bmod 2}{4k^2}\Big)\,\mathbb{P}\Big[v_i > \frac{\Delta}{k}\Big(1-\frac{\epsilon}{2}\Big) \text{ for all } i \in \{1,\dots,k\}\Big],\end{aligned} \tag{6.1}$$

where the probability is taken over partitions $\boldsymbol{v} \in P_k(\Delta)$ chosen uniformly at random.

We proceed to bound this probability from below. Each v_i is distributed according to a binomial distribution with mean $\frac{\Delta}{k}$ and variance $\frac{\Delta}{k}(1-\frac{1}{k})$, so by Chebychev's inequality we have for all $z > 0$ that

$$\mathbb{P}\left[\Big|v_i - \frac{\Delta}{k}\Big| \geq z\sqrt{\frac{\Delta}{k}\Big(1-\frac{1}{k}\Big)}\,\right] \leq \frac{1}{z^2}. \tag{6.2}$$

Thus

$$\mathbb{P}\left[v_i \leq \frac{\Delta}{k}\left(1-\frac{\epsilon}{2}\right)\right] \leq \mathbb{P}\left[\left|v_i - \frac{\Delta}{k}\right| \geq \frac{\Delta}{k}\cdot\frac{\epsilon}{2}\right]$$
$$\leq \mathbb{P}\left[\left|v_i - \frac{\Delta}{k}\right| \geq \frac{\Delta}{k}\cdot\frac{\epsilon}{2}\sqrt{1-\frac{1}{k}}\,\right] \leq \frac{4k}{\Delta\cdot\epsilon^2},$$

where the second inequality holds because $\sqrt{1-1/k} \leq 1$, and the third inequality follows from (6.2) by setting $z = \frac{\epsilon}{2}\sqrt{\Delta/k}$. By the union bound, and by the assumption that $\Delta \geq 8k^2/\epsilon^3$,

$$\mathbb{P}\left[v_i > \frac{\Delta}{k}\left(1-\frac{\epsilon}{2}\right) \text{ for all } i \in \{1,\dots,k\}\right] \geq 1 - \frac{4k^2}{\Delta\cdot\epsilon^2} \geq 1-\frac{\epsilon}{2}.$$

Combining this with (6.1), we obtain

$$\alpha_k(\Delta) \geq \left(1-\frac{\epsilon}{2}\right)^2\left(\frac{3}{4}-\frac{1}{2k}-\frac{k \bmod 2}{4k^2}\right) > (1-\epsilon)\left(\frac{3}{4}-\frac{1}{2k}-\frac{k \bmod 2}{4k^2}\right),$$

as claimed. □

To see that this bound cannot be improved significantly, consider situations where one agent, i^*, receives Δ nominations and the remaining agents receive $\Delta/2$ nominations. As the number of agents tends to infinity, the permutation mechanism selects i^* only if i^* receives at least $\Delta/2$ nominations from agents appearing before it in the permutation, which happens with probability at most $1/2$. Otherwise an agent with $\Delta/2$ nominations is selected, which leads to an performance guarantee of $\alpha \leq 3/4$. A similar but more involved example was also given by Bousquet et al. [4].

7. No Abstentions. Let us finally consider the interesting special case of graphs in which all vertices have outdegree at least 1, and to this end denote

$$\mathcal{G}^+ = \bigcup\nolimits_{n\in\mathbb{N}}\Big\{(N,E)\in\mathcal{G}_n : \min\nolimits_{i\in N} |\{(i,j)\in E : j \in N\}\}| \geq 1\Big\}.$$

This case models situations where abstentions are not allowed and in particular includes the setting of Holzman and Moulin [15], where every agent casts exactly one nomination.

In Section 3 we obtained the first non-trivial bound for the special case, by combining the simple observation that *any* mechanism is optimal on graphs in $\mathcal{G}^+$ with maximum indegree 1 and a new lower bound on the performance of the 2-partition mechanism for graphs with maximum indegree at least 2. Similar arguments can be applied to the k-partition and permutation mechanisms as well, and it is in fact not difficult to show that the permutation mechanism is 2/3-optimal on graphs in $\mathcal{G}^+$ with maximum indegree at most 2. By bounding its performance on graphs with maximum indegree 3 or more, we then obtain the following result.

THEOREM 7.1. *The permutation mechanism is* $67/108$*-optimal on* $\mathcal{G}^+$.

Proof. Consider any graph $G = (N,E) \in \mathcal{G}^+$ with at least two vertices. Note that G contains a directed cycle, such that for every permutation $(\pi_1,\dots,\pi_{|N|})$ of N there exist $s,t \in \{1,\dots,|N|\}$ with $s<t$ and $(\pi_s,\pi_t)\in E$. After considering π_t, the permutation mechanism will have chosen a candidate with indegree 1 or higher, and will thus *always* select a vertex with indegree at least 1.

If $\Delta(G) = 1$, then every vertex has indegree exactly 1 and any mechanism including the permutation mechanism is optimal. If $\Delta(G) = 2$, there exist vertices

$i^*, i_1, i_2 \in N$ with $i_1 \neq i_2$ and $(i_1, i^*), (i_2, i^*) \in E$. For any permutation in which i^* appears after both i_1 and i_2, i.e., with probability 1/3, the permutation mechanism thus selects a vertex with indegree 2. As it always selects a vertex with indegree at least 1, the expected indegree of the selected vertex is at least $2/3 > 67/108$. It remains to establish 67/108-optimality when $\Delta(G) \geq 3$. By the same argument as in the proof of Theorem 4, and by Lemma 4.3, it suffices to show that $\lim_{k\to\infty} \alpha_k(3) \geq 67/108$.

The proof of this latter result is quite technical, and it is deferred to Appendix B. Here we instead prove the weaker statement that $\lim_{k\to\infty} \alpha_k(2) \geq 7/12$, which will illustrate some of the main ideas used in the proof of the stronger result and implies that the permutation mechanism is 7/12-optimal on $\mathcal{G}^+$.

By Lemma 4.2,

$$\begin{aligned}
\alpha_k(2) &= \frac{1}{2k^2} \sum\nolimits_{\boldsymbol{v}\in P_k(2)} \binom{\Delta}{\boldsymbol{v}} \min\bigl\{\boldsymbol{W}^k \cdot \boldsymbol{v}\bigr\} \\
&= \frac{1}{2k^2} \sum\nolimits_{\boldsymbol{x}\in\{1,\ldots,k\}^2} \min\bigl\{\boldsymbol{W}^k \cdot (\boldsymbol{e}^{k,x_1} + \boldsymbol{e}^{k,x_2})\bigr\} \\
&= \frac{1}{2k^2} \sum\nolimits_{\boldsymbol{x}\in\{1,\ldots,k\}^2} \left(2 - \max\nolimits_{j=1,\ldots,k}\left\{\frac{j}{k}\bigl(\chi[x_1 \geq j] + \chi[x_2 \geq j]\bigr)\right\}\right) \\
&= 1 - \frac{1}{k^3} \sum\nolimits_{\boldsymbol{x}\in\{1,\ldots,k\}^2} \max\left\{\min\{\boldsymbol{x}\}, \frac{\max\{\boldsymbol{x}\}}{2}\right\} = 1 - \frac{\beta_k}{k^3},
\end{aligned}$$

where $\boldsymbol{e}^{k,i}$ again denotes the ith unit vector in k dimensions, and $\beta_k = \sum_{\boldsymbol{x}\in\{1,\ldots,k\}^2} \max\bigl\{\min\{\boldsymbol{x}\}, \max\{\boldsymbol{x}\}/2\bigr\}$. Grouping vectors $\boldsymbol{x} \in \{1,\ldots,k\}^2$ by the number of entries that are equal to k,

$$\begin{aligned}
\beta_k &= \beta_{k-1} + 2\sum\nolimits_{y\in\{1,\ldots,k-1\}} \max\left\{y, \frac{k}{2}\right\} + k \\
&= \beta_{k-1} + 2\left(\frac{k^2}{4} + \sum\nolimits_{y\in\{k/2+1,\ldots,k-1\}} y\right) + o(k^2) \\
&= \beta_{k-1} + \frac{5}{4}k^2 + o(k^2).
\end{aligned}$$

Since $\beta_0 = 0$,

$$\beta_k = \sum_{\ell=1}^{k} \left(\frac{5}{4}\ell^2 + o(\ell^2)\right) = \frac{5}{12}k^3 + o(k^3)$$

so

$$\alpha_k(2) = \frac{7}{12} - \frac{o(k^3)}{k^3},$$

and thus $\lim_{k\to\infty} \alpha_k(2) = 7/12$. $\square$

One may wonder whether the bound of $67/108 \approx 0.62$ is tight, for the permutation mechanism or even in general. We leave this as an open question, but conclude by giving upper bounds of 2/3 and 3/4, respectively, on possible values of α for the permutation mechanism and any impartial mechanism.

To see that the permutation mechanism cannot be more than 2/3-optimal, consider the graph of Figure 5. The unique vertex with indegree 3 in this graph is selected by the permutation mechanism if and only if it appears in the last two positions of

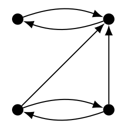

FIG. 5. *A graph on which the permutation mechanism is 2/3-optimal*

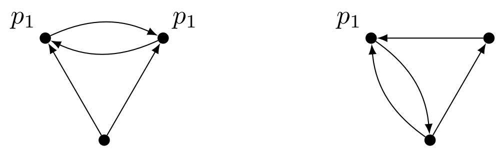

FIG. 6. *Impartial probability assignment for two graphs with $n = 3$*

the permutation, which happens with probability 1/2. Indeed, when it appears in one of the first two positions it has indegree at most 1 at the time it is considered by the mechanism. At the same time, one of the vertices in the last two positions has indegree 1 when it is considered and consequently gets selected. The expected indegree of the selected vertex is thus $3 \cdot 1/2 + 1 \cdot 1/2 = 2$, compared to a maximum indegree of 3. Interestingly this bound is attained for a graph with maximum indegree 3, which suggests that a matching lower bound may not be obtainable from a monotonicity result like that of Lemma 4.3.

The same upper bound of 2/3 holds asymptotically for the more restricted case considered by Holzman and Moulin [15], where every vertex has outdegree 1. To see this consider the graph $G = (N, E)$ with $N = \{1, \ldots, n\}$ and $E = \{(i, i+1) : i = 1, \ldots, n-2\} \cup \{(n-1, 1), (n, 1)\}$, and observe that the permutation mechanism selects vertex 1, the unique vertex with indegree 2, with significant probability only for permutations in which vertices $n-1$ and n both occur before 1. Since the latter happens with probability exactly 1/3, the expected indegree of the selected vertex is not significantly greater than $2 \cdot 1/3 + 1 \cdot 2/3 = 4/3$, compared to a maximum indegree of 2.

Our final result establishes upper bounds on α for *any* mechanism that is impartial and α-optimal on $\mathcal{G}^+$, and for different values of n. These bounds arise as dual solutions of an optimization problem characterizing the α-optimal impartial mechanisms for the maximum value of α. These dual solutions are optimal, and the upper bound therefore tight, for $n \leq 7$.

THEOREM 7.2. *Consider an impartial selection mechanism that is α-optimal on $\mathcal{G}_n^+$. Then*

$$\alpha \leq \begin{cases} 3/4 & \text{if } n = 3, \\ (3n-1)/4n & \text{otherwise.} \end{cases}$$

Proof. By Lemma 2.1 we can restrict our attention to symmetric mechanisms.

First assume that $n = 3$, and consider the two graphs shown in Figure 6. It is easily verified that any impartial mechanism must assign probabilities as shown, and it must therefore be the case that $p_1 \leq \frac{1}{2}$. In the graph on the right, the vertex assigned probability p_1 is the unique vertex with the maximum indegree of 2, and

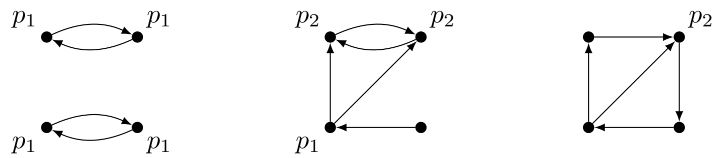

FIG. 7. *Impartial probability assignment for three graphs with $n = 4$*

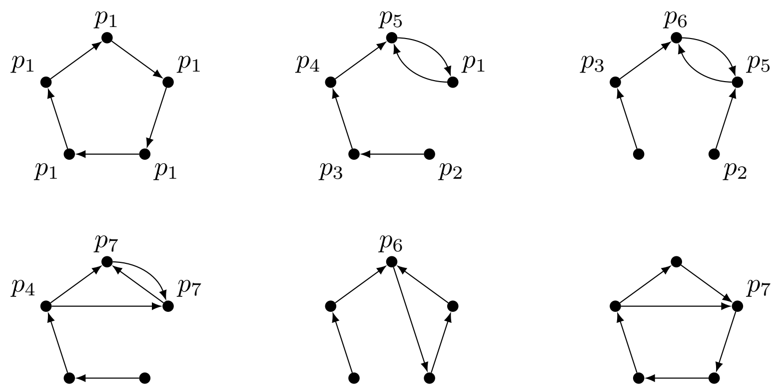

FIG. 8. *Impartial probability assignment for six graphs with $n = 5$*

thus

$$\alpha \leq \frac{2p_1 + (1 - p_1)}{2} = \frac{p_1 + 1}{2} \leq \frac{3}{4}.$$

Now assume that $n \geq 4$ even, and consider the set of three graphs on n vertices with edges among vertices 1 to 4 as in Figure 7 and the remaining $n - 4$ vertices grouped in pairs such that there is an edge from vertex $2i - i$ to vertex $2i$ and an edge from vertex $2i$ to vertex $2i - 1$. It is easily verified that any impartial mechanism must assign probabilities as in Figure 7, and thus $np_1 = 1$ and $p_1 + 2p_2 \leq 1$. Moreover, the vertex assigned probability p_2 in the rightmost graph is the unique vertex with indegree 2 in that graph, and thus

$$\alpha \leq \frac{2p_2 + (1 - p_2)}{2} = \frac{p_2 + 1}{2} \leq \frac{\frac{n-1}{2n} + 1}{2} = \frac{3n - 1}{4n}.$$

Now assume that $n = 5$, and consider the six graphs in Figure 8. It is easily verified that any impartial mechanism must assign probabilities as shown, so

$$p_1 = 1/5, \tag{7.1}$$

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1, \tag{7.2}$$

$$p_2 + p_3 + p_5 + p_6 \leq 1, \tag{7.3}$$

$$p_4 + 2p_7 \leq 1. \tag{7.4}$$

By adding (7.1), (7.3), and (7.4) and subtracting (7.2),

$$p_6 + 2p_7 \leq \frac{6}{5} \qquad \text{and thus} \qquad \min\{p_6, p_7\} \leq \frac{2}{5}.$$

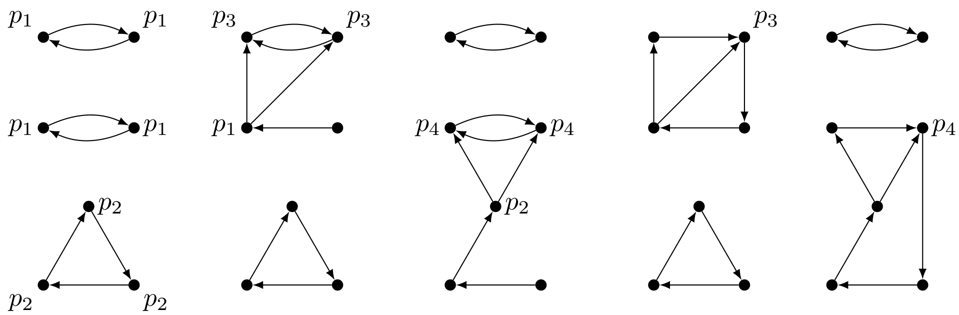

FIG. 9. *Impartial probability assignment for five graphs with $n = 7$*

The vertices assigned probabilities p_6 and p_7 in the two rightmost graphs in the bottom row of Figure 8 are the unique vertices with indegree 2 in those graphs, so

$$\alpha \leq \frac{2p_6 + (1 - p_6)}{2} = \frac{p_6 + 1}{2} \qquad \text{and} \qquad \alpha \leq \frac{2p_7 + (1 - p_7)}{2} = \frac{p_7 + 1}{2},$$

and thus

$$\alpha \leq \frac{\min\{p_6, p_7\} + 1}{2} \leq \frac{\frac{2}{5} + 1}{2} = \frac{7}{10} = \frac{3n - 1}{4n}.$$

Finally assume that $n \geq 7$ odd, and consider the set of five graphs on n vertices with edges among vertices 1 to 7 as in Figure 9 and the remaining $n - 7$ vertices grouped in pairs such that there is an edge from vertex $2i - i$ to vertex $2i$ and an edge from vertex $2i$ to vertex $2i - 1$. It is easily verified that any impartial mechanism must assign probabilities as in Figure 9, so

$$\begin{aligned} (n-3)p_1 + 3p_2 &= 1, \\ p_1 + 2p_3 &\leq 1, \\ p_2 + 2p_4 &\leq 1. \end{aligned}$$

The vertices assigned probabilities p_3 and p_4 in the two rightmost graphs are the unique vertices with indegree 2 in those graphs, so

$$\begin{aligned} \alpha &\leq \frac{2p_3 + (1 - p_3)}{2} = \frac{p_3 + 1}{2} \leq \frac{\frac{1 - p_1}{2} + 1}{2} = \frac{3 - p_1}{4}, \\ \alpha &\leq \frac{2p_4 + (1 - p_4)}{2} = \frac{p_4 + 1}{2} \leq \frac{\frac{1 - p_2}{2} + 1}{2} = \frac{3 - p_2}{4}, \end{aligned}$$

and thus

$$\alpha \leq \frac{3 - \max\{p_1, p_2\}}{4} \leq \frac{3 - \frac{1}{n}}{4} = \frac{3n - 1}{4n},$$

where the second inequality holds because $\max\{p_1, p_2\} \geq 1/n$. $\square$

Somewhat surprisingly, restricting the set of graphs even further, by requiring that every vertex has outdegree exactly 1, does not enable significantly better impartial mechanism. Using similar arguments as in the proof of Theorem 7.2, it can be shown that in this case any impartial and α-optimal mechanism must satisfy $\alpha \leq 5/6$ if $n = 3$, $\alpha \leq \frac{6n-1}{8n}$ if $n \geq 6$ and even, and $\alpha \leq \frac{3}{4}$ otherwise. These bounds are tight for $n \leq 9$.

Appendix A. Proof of Theorem 3.2. Using Lemma 3.1 it is straightforward to show that $\alpha_2(1) = 1/4$.

Now assume that Δ is strictly positive and even. Then, by Lemma 3.1,

$$\alpha_2(\Delta) = \frac{1}{\Delta 2^\Delta} \sum_{k=0}^{\frac{\Delta}{2}-1} \binom{\Delta}{k} \cdot k + \frac{1}{\Delta 2^\Delta} \binom{\Delta}{\Delta/2} \cdot \frac{\Delta}{2} + \frac{1}{\Delta 2^\Delta} \sum_{k=\frac{\Delta}{2}+1}^{\Delta} \binom{\Delta}{k} \cdot \frac{\Delta}{2}.$$

By symmetry of the binomial distribution with success probability $1/2$,

$$\frac{1}{2} \cdot \frac{1}{2^\Delta} \binom{\Delta}{\Delta/2} + \frac{1}{2^\Delta} \sum_{k=0}^{\frac{\Delta}{2}-1} \binom{\Delta}{k} = \frac{1}{2} \cdot \frac{1}{2^\Delta} \binom{\Delta}{\Delta/2} + \frac{1}{2^\Delta} \sum_{k=\frac{\Delta}{2}+1}^{\Delta} \binom{\Delta}{k} = \frac{1}{2}$$

and thus

$$\begin{aligned}
\alpha_2(\Delta) &= \frac{1}{\Delta 2^\Delta} \sum_{k=0}^{\frac{\Delta}{2}-1} \binom{\Delta}{k} \cdot k + \frac{1}{2} \cdot \frac{1}{\Delta 2^\Delta} \binom{\Delta}{\Delta/2} \cdot \frac{\Delta}{2} + \frac{1}{4} \\
&= \frac{1}{2^\Delta} \sum_{k=1}^{\frac{\Delta}{2}-1} \frac{(\Delta-1)!}{(\Delta-k)!(k-1)!} + \frac{1}{2^{\Delta+2}} \binom{\Delta}{\Delta/2} + \frac{1}{4} \\
&= \frac{1}{2^\Delta} \sum_{k=1}^{\frac{\Delta}{2}-1} \binom{\Delta-1}{k-1} + \frac{1}{2^{\Delta+2}} \binom{\Delta}{\Delta/2} + \frac{1}{4} \\
&= \frac{1}{2} \cdot \frac{1}{2^{\Delta-1}} \sum_{k=0}^{\frac{\Delta}{2}-2} \binom{\Delta-1}{k} + \frac{1}{2^{\Delta+2}} \binom{\Delta}{\Delta/2} + \frac{1}{4}.
\end{aligned}$$

Also by symmetry of the binomial distribution, $\frac{1}{2^{\Delta-1}} \sum_{k=0}^{\frac{\Delta}{2}-1} \binom{\Delta-1}{k} = \frac{1}{2}$, and thus

$$\begin{aligned}
\alpha_2(\Delta) &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^{\Delta-1}} \binom{\Delta-1}{\Delta/2-1}\right) + \frac{1}{2^{\Delta+2}} \binom{\Delta}{\Delta/2} + \frac{1}{4} \\
&= \frac{1}{2} - \frac{1}{2^\Delta} \binom{\Delta-1}{\Delta/2-1} + \frac{1}{2^{\Delta+2}} \binom{\Delta}{\Delta/2}.
\end{aligned}$$

Since $\Delta - 1$ is odd, $\binom{\Delta-1}{\Delta/2} = \binom{\Delta-1}{\Delta/2-1}$ and thus $\binom{\Delta}{\Delta/2} = \binom{\Delta-1}{\Delta/2} + \binom{\Delta-1}{\Delta/2-1} = 2\binom{\Delta-1}{\Delta/2-1}$. We conclude that

$$\alpha_2(\Delta) = \frac{1}{2} - \frac{1}{2^{\Delta+1}} \binom{\Delta}{\Delta/2} + \frac{1}{2^{\Delta+2}} \binom{\Delta}{\Delta/2} = \frac{1}{2} - \frac{1}{2^{\Delta+2}} \binom{\Delta}{\Delta/2},$$

as claimed.

Finally assume that $\Delta \geq 3$ and odd. Then, by Lemma 3.1,

$$\alpha_2(\Delta) = \frac{1}{\Delta 2^\Delta} \sum_{k=0}^{\lfloor \frac{\Delta}{2} \rfloor} \binom{\Delta}{k} \cdot k + \frac{1}{\Delta 2^\Delta} \sum_{k=\lceil \frac{\Delta}{2} \rceil}^{\Delta} \binom{\Delta}{k} \cdot \frac{\Delta}{2}.$$

By symmetry of the binomial distribution, $\frac{1}{2^\Delta}\sum_{k=\lceil \frac{\Delta}{2}\rceil}^{\Delta}\binom{\Delta}{k}=\frac{1}{2}$ and thus

$$\begin{aligned}\alpha_2(\Delta) &= \frac{1}{\Delta 2^\Delta}\sum_{k=0}^{\frac{\Delta-1}{2}}\binom{\Delta}{k}\cdot k+\frac{1}{4}\\ &= \frac{1}{2^\Delta}\sum_{k=1}^{\frac{\Delta-1}{2}}\frac{(\Delta-1)!}{(\Delta-k)!(k-1)!}+\frac{1}{4}\\ &= \frac{1}{2^\Delta}\sum_{k=1}^{\frac{\Delta-1}{2}}\binom{\Delta-1}{k-1}+\frac{1}{4}\\ &= \frac{1}{2}\cdot\frac{1}{2^{\Delta-1}}\sum_{k=0}^{\frac{\Delta-1}{2}-1}\binom{\Delta-1}{k}+\frac{1}{4}.\end{aligned}$$

Since $\Delta-1$ is even, and again using symmetry of the binomial distribution,

$$\frac{1}{2^{\Delta-1}}\sum_{k=0}^{\frac{\Delta-1}{2}-1}\binom{\Delta-1}{k}+\frac{1}{2}\cdot\frac{1}{2^{\Delta-1}}\binom{\Delta-1}{(\Delta-1)/2}=\frac{1}{2}.$$

We conclude that

$$\alpha_2(\Delta)=\frac{1}{4}+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2^\Delta}\binom{\Delta-1}{(\Delta-1)/2}\right)=\alpha_2(\Delta-1),$$

as claimed.

Appendix B. Improved Bound in the Absence of Abstentions. The following technical lemma completes the proof of Theorem 7.1.

LEMMA B.1. *For α_k as in Lemma 4.2,* $\lim_{k\to\infty}\alpha_k(3)\geq 67/108$.

Proof. By Lemma 4.2,

$$\begin{aligned}\alpha_k(3) &= \frac{1}{3k^3}\sum\nolimits_{\boldsymbol{v}\in P_k(3)}\binom{\Delta}{\boldsymbol{v}}\min\{\boldsymbol{W}^k\cdot\boldsymbol{v}\}\\ &= \frac{1}{3k^3}\sum\nolimits_{\boldsymbol{x}\in\{1,\dots,k\}^3}\min\bigl\{\boldsymbol{W}^k\cdot(\boldsymbol{e}^{k,x_1}+\boldsymbol{e}^{k,x_2}+\boldsymbol{e}^{k,x_3})\bigr\}\\ &= \frac{1}{3k^3}\sum\nolimits_{\boldsymbol{x}\in\{1,\dots,k\}^3}\left(3-\max\nolimits_{j=1,\dots,k}\left\{\frac{j}{k}\bigl(\chi[x_1\geq j]+\chi[x_2\geq j]+\chi[x_3\geq j]\bigr)\right\}\right).\end{aligned}$$

For $\boldsymbol{x}\in\{1,\dots,k\}^3$, denote by $\bar{\boldsymbol{x}}=(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ the vector that contains the entries of $\boldsymbol{x}$ in non-decreasing order. Then

$$\alpha_k(3)=1-\frac{1}{3k^4}\sum\nolimits_{\boldsymbol{x}\in\{1,\dots,k\}^3}\max\bigl\{3\bar{x}_1,2\bar{x}_2,\bar{x}_3\bigr\}=1-\frac{\beta_k}{3k^4},$$

where $\beta_k=\sum_{\boldsymbol{x}\in\{1,\dots,k\}^3}\max\{3\bar{x}_1,2\bar{x}_2,\bar{x}_3\}$, and we are interested in bounding β_k from above. Grouping vectors $\boldsymbol{x}\in\{1,\dots,k\}^3$ by the number of entries that are equal to k,

$$\begin{aligned}\beta_k &= \beta_{k-1} + 3\sum\nolimits_{\boldsymbol{x}\in\{1,\dots,k-1\}^2} \max\{3\bar{x}_1, 2\bar{x}_2, k\} \\ &\qquad\qquad\qquad\qquad + 3\sum\nolimits_{y\in\{1,\dots,k-1\}} \max\{3y, 2k\} + 3k \\ &= \beta_{k-1} + 6\sum_{y=1}^{k-1}\sum_{z=y+1}^{k-1}\max\{3y, 2z, k\} + \mathcal{O}(k^2) \\ &\le \beta_{k-1} + 6\sum_{y=1}^{\lfloor\frac{k}{3}\rfloor}\sum_{z=y+1}^{k-1}\max\{2z, k\} + 6\sum_{y=\lceil\frac{k}{3}\rceil}^{k-1}\sum_{z=y+1}^{k-1}\max\{3y, 2z\} + \mathcal{O}(k^2). \end{aligned} \tag{B.1}$$

The last inequality arises because $y = k/3$ appears in both sums when k is divisible by 3. We now claim that for all $k \in \mathbb{N}$,

$$\sum_{y=1}^{\lfloor\frac{k}{3}\rfloor}\sum_{z=y+1}^{k-1}\max\{2z, k\} \le \frac{13}{36}k^3 + \mathcal{O}(k^2) \tag{B.2}$$

and

$$\sum_{y=\lceil\frac{k}{3}\rceil}^{k-1}\sum_{z=y+1}^{k-1}\max\{3y, 2z\} \le \frac{43}{108}k^3 + \mathcal{O}(k^2). \tag{B.3}$$

Before verifying that these claims are indeed correct, we show that they imply the desired result. By substituting (B.2) and (B.3) into (B.1),

$$\beta_k \le \beta_{k-1} + \frac{13}{6}k^3 + \frac{43}{18}k^3 + \mathcal{O}(k^2) = \frac{41}{9}k^3 + \mathcal{O}(k^2).$$

Since $\beta_0 = 0$,

$$\beta_k \le \sum_{\ell=1}^{k}\left(\frac{41}{9}\ell^3 + \mathcal{O}(k^2)\right) = \frac{41}{36}k^4 + \mathcal{O}(k^3)$$

and thus

$$\lim_{k\to\infty}\alpha_k(3) \ge 1 - \lim_{k\to\infty}\frac{\beta_k}{3k^4} = \frac{67}{108}.$$

It remains to show correctness of (B.2) and (B.3). For (B.2) we obtain

$$\begin{aligned}\sum_{y=1}^{\lfloor\frac{k}{3}\rfloor}\sum_{z=y+1}^{k-1}\max\{2z, k\} &= \sum_{z=2}^{\lfloor\frac{k}{3}\rfloor}(z-1)\max\{2z, k\} + \left\lfloor\frac{k}{3}\right\rfloor\sum_{z=\lfloor\frac{k}{3}\rfloor+1}^{k-1}\max\{2z, k\} \\ &\le \sum_{z=2}^{\lfloor\frac{k}{3}\rfloor}(z-1)k + \left\lfloor\frac{k}{3}\right\rfloor\sum_{z=\lfloor\frac{k}{3}\rfloor+1}^{\lfloor\frac{k}{2}\rfloor}k + 2\left\lfloor\frac{k}{3}\right\rfloor\sum_{z=\lceil\frac{k}{2}\rceil}^{k-1}z \\ &= k\left(\frac{k^2}{18} + \mathcal{O}(k)\right) + \frac{k^2}{3}\left(\frac{k}{6} + \mathcal{O}(1)\right) + \frac{2k}{3}\left(\frac{3k^2}{8} + \mathcal{O}(k)\right) \\ &= \frac{13}{36}k^3 + \mathcal{O}(k^2),\end{aligned}$$

as claimed.

For (B.3), observe that

$$\sum_{y=\lceil \frac{k}{3}\rceil}^{k-1}\sum_{z=y+1}^{k-1}\max\{3y,2z\} \leq \sum_{y=\lceil \frac{k}{3}\rceil}^{\lfloor \frac{2k}{3}\rfloor}\sum_{z=y+1}^{\lfloor \frac{3y}{2}\rfloor} 3y + \sum_{y=\lceil \frac{k}{3}\rceil}^{\lfloor \frac{2k}{3}\rfloor}\sum_{z=\lceil \frac{3y}{2}\rceil}^{k-1} 2z + \sum_{y=\lceil \frac{2k}{3}\rceil}^{k-1}\sum_{z=y+1}^{k-1} 3y$$

$$= \sum_{y=\lceil \frac{k}{3}\rceil}^{\lfloor \frac{2k}{3}\rfloor} 3y\left\lfloor \frac{y}{2}\right\rfloor + \sum_{y=\lceil \frac{k}{3}\rceil}^{\lfloor \frac{2k}{3}\rfloor}\sum_{z=\lceil \frac{3y}{2}\rceil}^{k-1} 2z + \sum_{y=\lceil \frac{2k}{3}\rceil}^{k-1} 3y(k-1-y). \tag{B.4}$$

For the second sum,

$$\begin{aligned}
\sum_{y=\lceil \frac{k}{3}\rceil}^{\lfloor \frac{2k}{3}\rfloor}\sum_{z=\lceil \frac{3y}{2}\rceil}^{k-1} 2z &= \sum_{z=\lceil \frac{k}{2}\rceil}^{k-1} 2z \cdot \left|\left\{ y \in \left\{\left\lceil \frac{k}{3}\right\rceil, \dots, \left\lfloor \frac{2k}{3}\right\rfloor\right\} : \left\lceil \frac{3y}{2}\right\rceil \leq z\right\}\right| \\
&\leq \sum_{z=\lceil \frac{k}{2}\rceil}^{k-1} 2z \cdot \left|\left\{ y \in \left\{\left\lceil \frac{k}{3}\right\rceil, \dots, \left\lfloor \frac{2k}{3}\right\rfloor\right\} : y \leq \frac{2z}{3}\right\}\right| \\
&= \sum_{z=\lceil \frac{k}{2}\rceil}^{k-1} 2z \cdot \left|\left\{ y \in \left\{\left\lceil \frac{k}{3}\right\rceil, \dots, \left\lfloor \frac{2z}{3}\right\rfloor\right\}\right\}\right| \\
&\leq \sum_{z=\lceil \frac{k}{2}\rceil}^{k-1} 2z \cdot \left(\frac{2z}{3} - \frac{k}{3}\right),
\end{aligned}$$

and by substituting this into (B.4),

$$\begin{aligned}
\sum_{y=\lceil \frac{k}{3}\rceil}^{k-1}\sum_{z=x+1}^{k-1}\max\{3y,2z\} &\leq \sum_{y=\lceil \frac{k}{3}\rceil}^{\lfloor \frac{2k}{3}\rfloor} 3y\left\lfloor \frac{y}{2}\right\rfloor + \sum_{z=\lceil \frac{k}{2}\rceil}^{k-1} 2z\left(\frac{2z}{3} - \frac{k}{3}\right) + \sum_{y=\lceil \frac{2k}{3}\rceil}^{k-1} 3y\big(k-1-y\big) \\
&\leq \sum_{y=\lceil \frac{k}{3}\rceil}^{\lfloor \frac{2k}{3}\rfloor} \frac{3y^2}{2} + \sum_{z=\lceil \frac{k}{2}\rceil}^{k-1}\left(\frac{4z^2}{3} - \frac{2kz}{3}\right) + \sum_{y=\lceil \frac{2k}{3}\rceil}^{k-1}\big(3ky - 3y^2\big).
\end{aligned}$$

Using multiple times that $\sum_{y=1}^{\ell} y = \ell^2/2 + \mathcal{O}(\ell)$ and $\sum_{y=1}^{\ell} y^2 = \ell^3/3 + \mathcal{O}(\ell^2)$ we obtain

$$\begin{aligned}
\sum_{y=\lceil \frac{k}{3}\rceil}^{k-1}\sum_{z=y+1}^{k-1}\max\{3y,2z\} &\leq \frac{3}{2}\cdot\frac{1}{3}\left(\frac{8k^3}{27} - \frac{k^3}{27}\right) + \frac{4}{3}\cdot\frac{1}{3}\left(k^3 - \frac{k^3}{8}\right) - \frac{2}{3}\cdot\frac{k}{2}\left(k^2 - \frac{k^2}{4}\right) \\
&\quad + 3\cdot\frac{k}{2}\left(k^2 - \frac{4k^2}{9}\right) - 3\cdot\frac{1}{3}\left(k^3 - \frac{8k^3}{27}\right) + \mathcal{O}(k^2) \\
&= \frac{43}{108}k^3 + \mathcal{O}(k^2),
\end{aligned}$$

which shows (B.3). $\square$

REFERENCES

[1] N. Alon and M. Naor. Coin-flipping games immune against linear-sized coalitions. *SIAM Journal on Computing*, 22(2):403–417, 1993.

[2] N. Alon, F. Fischer, A. D. Procaccia, and M. Tennenholtz. Sum of us: Strategyproof selection from the selectors. In *Proceedings of the 13th Conference on Theoretical Aspects of Rationality and Knowledge*, pages 101–110, 2011.

[3] S. Antonakopoulos. Fast leader-election protocols with bounded cheaters' edge. In *Proceedings of the 38th Annual ACM Symposium on Theory of Computing*, pages 187–196, 2006.

[4] N. Bousquet, S. Norin, and A. Vetta. A near-optimal mechanism for impartial selection. In *Proceedings of the 10th International Conference on Web and Internet Economics*, pages 133–146, 2014.

[5] J. Cooper and N. Linial. Fast perfect-information leader-election protocols with linear immunity. *Combinatorica*, 15(3):319–332, 1995.

[6] G. de Clippel, H. Moulin, and N. Tideman. Impartial division of a dollar. *Journal of Economic Theory*, 139(1):176–191, 2008.

[7] O. Dekel, F. Fischer, and A. D. Procaccia. Incentive compatible regression learning. *Journal of Computer and System Sciences*, 76(8):759–777, 2010.

[8] S. Dobzinski, N. Nisan, and M. Schapira. Truthful randomized mechanisms for combinatorial auctions. In *Proceedings of the 38th Annual ACM Symposium on Theory of Computing*, pages 644–652, 2006.

[9] U. Feige. Noncryptographic selection protocols. In *Proceedings of the 40th Symposium on Foundations of Computer Science*, pages 142–152, 1999.

[10] U. Feige, A. Flaxman, J. D. Hartline, and R. Kleinberg. On the competitive ratio of the random sampling auction. In *Proceedings of the 1st International Workshop on Internet and Network Economics*, pages 878–886, 2005.

[11] A. Fiat, A. V. Goldberg, J. D. Hartline, and A. R. Karlin. Competitive generalized auctions. In *Proceedings of the 34th Annual ACM Symposium on Theory of Computing*, pages 72–81, 2002.

[12] F. Fischer and M. Klimm. Optimal impartial selection. In *Proceedings of the 15th ACM Conference on Economics and Computation*, pages 803–820, 2014.

[13] E. Friedman, P. Resnick, and R. Sami. Manipulation-resistant reputation systems. In N. Nisan, T. Roughgarden, É. Tardos, and V. Vazirani, editors, *Algorithmic Game Theory*, chapter 27, pages 677–697. Cambridge University Press, 2007.

[14] A. V. Goldberg, J. D. Hartline, A. R. Karlin, M. Saks, and A. Wright. Competitive auctions. *Games and Economic Behavior*, 55:242–269, 2006.

[15] R. Holzman and H. Moulin. Impartial nominations for a prize. *Econometrica*, 81(1): 173–196, 2013.

[16] M. R. Merrifield and D. G. Saari. Telescope time without tears: A distributed approach to peer review. *Astronomy & Geophysics*, 50(4):16–20, 2009.

[17] A. D. Procaccia and M. Tennenholtz. Approximate mechanism design without money. *ACM Transactions on Economics and Computation*, 1(4):Article 18, 2013.

[18] J. Witkowski and D. C. Parkes. Peer prediction without a common prior. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, pages 964–981, 2012.