Non-Truthful Position Auctions Are More Robust to Misspecification*

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Abstract

We consider the classical model of sponsored search due to Edelman et al. and Varian, and examine how robust standard position auctions are to a misspecification of the position-dependent quality factors used by this model. We show that under both complete and incomplete information a non-truthful position auction admits an efficient equilibrium for a strictly broader range of parameter values than the Vickrey-Clarke-Groves (VCG) mechanism, which would be truthful if the parameters were specified correctly. Our result for complete information concerns the generalized second-price (GSP) mechanism, and is driven by a detailed understanding of the Nash equilibrium polytopes of the VCG mechanism and the GSP mechanism. Our result for incomplete information concerns the generalized first-price (GFP) mechanism, and uses a surprising connection between the unique candidate equilibrium bidding functions of the VCG mechanism and the GFP mechanism.

1 Introduction

Online advertising is the main source of revenue for technology companies such as Google or Facebook [e.g., 2]. A particularly important form of online advertising is sponsored search, where sponsored results are shown alongside organic results of a web search engine. After going through a number of evolutionary steps, sponsored search converged to a system in which ad slots for each individual search are sold through an auction (see, e.g., the article of Edelman et al. [19] for a brief history of sponsored search). Sponsored search auctions differ in a number of ways from the auctions traditionally studied in auction theory. A particular

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distinctive feature is the extremely high rate at which auctions are conducted, which in turn mandates an auction design that minimizes informational and computational requirements.

The key properties of practical sponsored search auctions are captured by the canonical model of Edelman et al. [19] and Varian [34]. In this model, n bidders interested in placing an ad compete for k positions in which an ad could be placed. Each bidder comes with a value v_i and each position with a quality β_j, where $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_k$. The value that bidder i derives from position j is $\beta_j \cdot v_i$.[1] In order to determine an allocation of bidders to positions along with payments the bidders have to make, each bidder i submits a bid b_i. Then bidders are ranked from high to low, so that $b_1 \geq b_2 \geq \cdots \geq b_n$, and bidder i is assigned position i for $1 \leq i \leq k$. The most common pricing rule used in practice is that of the so-called generalized second-price (GSP) mechanism, in which bidder i pays an amount equal to $\beta_i \cdot b_{i+1}$. An alternative, which was used in the early days of sponsored search and is again gaining traction, is the generalized first-price (GFP) mechanism, in which bidder i pays $\beta_i \cdot b_i$. The industry standard for implementing these mechanisms is to charge advertisers whenever a user clicks on their ads.

It is worth pointing out that this design minimizes the amount of information that bidders have to transfer to the auctioneer, and the computational overhead of both bidders and auctioneer: each bidder transfers only a single number, the allocation can be determined using a simple greedy rule, and the payments depend on the allocation and bids in a straightforward manner.

The crucial assumption underlying the design is that bidders have single-dimensional valuations, i.e., that values can indeed be written as $\beta_j \cdot v_i$. This assumption is backed up through extensive experiments [e.g., 34], but auction specifications also suggest that β_j is more than just the total number of clicks a position receives or, after normalization, the click-through rate of a position [e.g., 30, 21]. Indeed, search engine providers have developed sophisticated machine learning techniques to learn these quality factors from data [22, 29]. This has important implications for practical designs that charge users per click. For example, to achieve a payment of $\beta_i \cdot b_{i+1}$ as in the GSP mechanism for a position that receives γ_i clicks, the search engine has to charge $\beta_i \cdot b_{i+1}/\gamma_j$ per click.

An interesting feature of the designs used in practice, the GSP mechanism and the GFP mechanism, is their lack of truthfulness, meaning that bidders can benefit from misreporting their bids (see the article of Edelman et al. [19] for a simple example). These designs are thus commonly analyzed in the Nash equilibria of a complete-information environment, or in the Bayes-Nash equilibria of an incomplete-information environment. A truthful design does exist: the Vickrey-Clarke-Groves (VCG) mechanism would assign bidders to positions in the same way as the GSP or GFP mechanisms, but charge bidder i their externality, which is equal to $\sum_{j=i}^{k}(\beta_j - \beta_{j+1}) \cdot b_{j+1}$. Apart perhaps from the slightly more complex payment rule, the VCG mechanism may thus seem preferable to the GSP and GFP mechanisms in the context of sponsored search. Indeed, most of the "common complaints" regarding the

[1] Edelman et al. [19, Section III] discuss a generalization of this model with bidder- as well as position-specific effects. When the two effects are separable the bidder-specific part can be folded into the valuations, and Edelman et al. provide evidence that near separability holds in practice.

VCG mechanism only apply to multi-dimensional settings [3, 33], and like the GSP and GFP mechanisms it can also be implemented in a pay-per-click fashion [35].

1.1 Model and research question

The vast majority of the literature on sponsored search auctions has focused on the case where the quality factors $\beta_1, \ldots, \beta_k$ are known to the auctioneer, and has tried to understand the equilibria that respectively arise for the GSP mechanism [e.g., 19, 34, 20, 9]) and the GFP mechanism [e.g., 18, 12, 24, 15]). There are, however, a number of reasons why in practice the quality factors used by the search engine may differ from the truth. Indeed, machine learning is capable of producing very accurate estimates, but it will typically not produce an estimate that is entirely exact. The true quality factors may also shift over time, either in the form of slow drifts or shocks due to unforeseen events. There are, finally, compelling reasons why a search engine may be unable to even observe the true quality factors. Milgrom [31] provides a simple example with two types of users of a search engine that leads to a difference between click-through rates, observed by the search engine, and conversion rates, in which bidders are interested. In practice similar discrepancies are likely to arise for more complicated reasons, for example when learning by the search engine takes places on aggregate over a large set of users while each bidder is interested in targeting only a subset of the users. A certain degree of robustness to a slight misspecification of the quality factors, and the resulting misspecification of the bidding language, is thus a desirable property of an auction design.

We will study this property using a stylized model that adds a small variation to the standard model of Edelman et al. and Varian. In our model the true quality factors are $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_k$, so that the value of bidder i for position j is $\alpha_j \cdot v_i$, whereas the auctioneer uses qualities $\beta_1 \geq \beta_2 \geq ... \geq \beta_k$ when computing the payments. In the GSP mechanism with bids $b_1 \geq b_2 \geq \cdots \geq b_n$, for example, advertiser i would thus be assigned position i, receive a value of $\alpha_i \cdot v_i$, and pay an amount equal to $\beta_i \cdot b_{i+1}$.[2]

Implicit in this model is an assumption that the bidders know the true quality factors β_j while the search engine uses imprecise estimates α_j. This seems at odds with a perceived informational asymmetry between search engine and bidders, which would place the search engine at a clear advantage. However, as we have pointed out, practical and theoretical evidence suggests that the true quality factors cannot in general be observed and learned completely accurately by a search engine. Bidders on the other hand do not have explicit access to the relative values of the positions, but they can observe their own utility and can thus learn how to bid through repeated interaction with the mechanism. Our static model, just as the original model of Edelman et al. [19] and Varian [34], captures the steady states of such game-playing dynamics.

It is natural to ask how the standard position auction designs perform in the more general model, and how they compare in terms of their robustness to a slight misspecification of the

[2] The arguments of Edelman et al. concerning bidder-specific effects apply in the same way to our more general model, so there is no need to model this type of misspecification explicitly.

	complete	incomplete symmetric
GSP	$\mathcal{I}_{\text{GSP}}$	$\emptyset$
	$\cup$	
VCG	$\mathcal{I}_{\text{VCG}}$	$\mathcal{I}_{\text{VCG}}$
		$\cap$
GFP	$\emptyset$	$\mathcal{I}_{\text{GFP}}$

Figure 1: Overview of our results. Under complete information the set of instances $\mathcal{I}_{\text{GSP}}$ for which GSP admits an efficient equilibrium is a superset of instances $\mathcal{I}_{\text{VCG}}$ for which the VCG mechanism admits an efficient equilibrium. An analog result applies to GFP and VCG under incomplete information with symmetric distributions. A $\emptyset$ symbol indicates that the respective mechanism may not possess an equilibrium, even when $\alpha = \beta$.

bidding language. While the VCG mechanism ceases to be truthful, for the simple reason that truthful bidding is no longer possible, one may hope for it to remain preferable to the GSP and GFP mechanisms. Indeed, the VCG mechanism uses a payment rule designed to align the interests of the bidders and the auctioneer, and it may continue to do so approximately when the bidding language is misspecified slightly.

We will specifically compare the robustness of different auction mechanisms to a misspecified bidding language by characterizing for which pairs of α_j and β_j each mechanism enables the existence of an efficient equilibrium. Two strong arguments exist that support this positivist approach. First, we can think of the existence of an efficient equilibrium as a necessary condition for a good design, in the sense that bidders should at least in principle be able to reach an outcome in which welfare is maximized. Second, simple equilibrium refinements such as envy-freeness imply that an efficient equilibrium will be selected if it exists [19, 34], and experimental evidence suggests that these are the equilibria that arise in practice [34]. We will compare the standard designs both in the complete information model in which sponsored search auctions are typically analyzed, and in the more classical model of auction theory in which bidders have incomplete information.

1.2 Results

We show that under both complete and incomplete information, a mechanism designed without the requirement of truthfulness is able to support an efficient outcome in equilibrium for a strictly larger set of values of α and β than the VCG mechanism. An overview of our results can be found in Figure 1. Failure of the VCG mechanism to produce an efficient outcome can in fact occur already when α is very close to β.

Complete Information We begin by considering settings with complete information, see Section 3. In these settings the GFP mechanism may not possess an equilibrium, even when $\alpha = \beta$ [19, 18]. We thus focus on the comparison between the VCG mechanism and the GSP mechanism. We show that for every instance with n bidders and k positions, and every pair of α and β for which the VCG mechanism possesses an efficient Nash equilibrium, the GSP mechanism does as well (Theorem 1). We also give examples that show that this inclusion is strict (Section 3.1).

Our proof is based on a detailed understanding of the respective equilibrium polytopes of the VCG mechanism and the GSP mechanism. We show that non-emptiness of the equilibrium polytope of the VCG mechanism implies the existence of a specific point within this polytope, which is then mapped to a specific point in the equilibrium polytope of the GSP mechanism to show non-emptiness of the latter. The specific point is one that satisfies the stronger requirement of envy-freeness, and can be reproduced with the same assignment and payments in the GSP mechanism.

Incomplete Information In Section 4 we consider settings with incomplete information and symmetric distributions. In these settings the GSP mechanism may not possess a Bayes-Nash equilibrium, even when $\alpha = \beta$ [20]. We thus compare the VCG mechanism to the GFP mechanism, and show a result analogous to that for complete information settings. Specifically, for every symmetric instance with n bidders and k positions, and every pair of α and β for which the VCG mechanism possesses an efficient Bayes-Nash equilibrium, the GFP mechanism does as well (Theorem 2), and this inclusion is strict (Section 4.1).

The proof of this result is rather intricate, and is driven by a surprising connection between the equilibrium bids in the VCG mechanism and those in the GFP mechanism. We begin by using a standard technique for equilibrium characterization that equates the expected payments in an efficient equilibrium as given by Myerson [32] with the respective payments in the two mechanisms. This yields a candidate equilibrium bidding function for each of the two mechanisms, and each of these functions constitutes an equilibrium if and only if it is strictly increasing almost everywhere. In the case of the VCG mechanism we encounter an ordinary differential equation, which we solve by appealing to a combinatorial equivalence. Even with the bidding functions for both the GFP mechanism and the VCG mechanism at hand, it is not trivial to show that the former is increasing for a larger set of values of α and β. To show that this is indeed the case, we exploit that the two bidding functions can respectively be written as $A(v)/B(v)$ and $A'(v)/B'(b)$, where A' and B' are the derivatives of A and B with respect to v.

It is known that for certain asymmetric settings the first-price auction may not possess an efficient equilibrium [28]. Together with our result this shows that in the general case, and with regard to efficient equilibria, neither mechanism dominates the other.

1.3 Related Work

A common aphorism in statistics, first formulated in this form by George Box [8], is that "all models are wrong." To Box, the interesting question was not whether a model is an exact representation of the real world, but whether it is close enough to the truth to be useful.

The role of model misspecification and model uncertainty in mechanism design has been highlighted by a number of authors [e.g., 6, 27, 11, 10, 23, 16, 17, 5]. We contribute to this line of work by investigating how robust standard position auctions are towards a misspecification of the bidding language. We also provide an alternative to the max-min approach employed in all of the articles given above. Other work has considered the design choice of the sponsored search industry of using a bidding language that restricts each bidder to submit a single number, and emphasized the advantages and possible risks associated with this choice [1, 7, 4, 31, 14]. However, none of this work enables a ranking of standard auction formats with respect to their robustness towards misspecification as we establish it here.

Our work highlights one advantage of non-truthful mechanisms for position auctions. A concurrent line of work has identified additional advantages of non-revelation mechanisms, such as amenability to statistical inference [13], and guaranteed revenue in dynamic settings [24] or across complete- and incomplete-information environments [15].

1.4 Open Problems

We compare standard position auction formats with respect to their ability to support an efficient equilibrium, when position-specific quality factors may be slightly misspecified. An interesting open question is to extend the analysis to approximately efficient equilibria in general incomplete-information environments with asymmetric distributions, where our comparison via efficient equilibria remains inconclusive.

2 Preliminaries

We study the standard setting of position auctions with k *positions* ordered by quality and $n \geq k$ *bidders* with *unit demand* and *single-dimensional valuations* for the positions.[3] Denote by $\mathbb{R}^k_{\geq} = \{x \in \mathbb{R}^k : x_j > 0, x_j \geq x_{j'} \text{ if } j < j'\}$ the set of k-dimensional vectors whose entries are positive and non-increasing. Given $\beta \in \mathbb{R}^k_{\geq}$, which we assume to be common knowledge among the bidders, the valuation of a particular bidder i can then be represented by a scalar $v_i \in \mathbb{R}$, such that $\beta_j v_i \geq 0$ is the bidder's value for position j. We will use the notational convention that $\beta_j = 0$ when $j > k$.

A *mechanism* in this setting receives a profile $b \in \mathbb{R}^n$ of bids, assigns positions to bidders in a one-to-one fashion, and charges each bidder a non-negative payment. It can be represented by a pair (g, p) of an *allocation rule* $g : \mathbb{R}^n \to S_n$ and a *payment rule* $p : \mathbb{R}^n \to \mathbb{R}^n$, such that for each $i \in \{1, \dots, n\}$, $g_i(b) = j$ for $j \in \{1, \dots, k\}$ means that bidder i is assigned position j and $p_i(b)$ is the payment charged to bidder i. We will be concerned exclusively with mechanisms that assign positions in non-increasing order of bids, and henceforth denote by g an allocation rule that does so and breaks ties in an arbitrary but consistent manner. The role of payments is to incentivize bids resulting in an *efficient assignment*, i.e., one where positions are assigned in order of valuations and *social welfare* $\sum_{i=1}^n \beta_{g_i(b)} v_i$ is maximized.

In reasoning about strategic behavior we make the usual assumption of quasi-linear preferences and consider two different models of information regarding the preferences of other

[3] The assumption that $n \geq k$ is without loss of generality, as in a setting with $n < k$ bidders the $k - n$ lowest-valued positions would never be assigned by the mechanisms we consider.

bidders. Under quasi-linear preferences, the *utility* $u_i(b, v_i)$ of bidder i with value v_i, in a given mechanism and for a given bid profile b, is equal to its valuation for the position it is assigned minus its payment, i.e., $u_i(b, v_i) = \beta_{g_i(b)} v_i - p_i(b)$. In the *complete information* model the values v_i are common knowledge among the bidders. A bid profile b is a *Nash equilibrium* of a given mechanism if no bidder has an incentive to change its bid assuming that the other bidders don't change their bids, i.e., if for every $i \in N$,

$$u_i\big(b, v_i\big) = \max_{x \in \mathbb{R}} u_i\big((b_{-i}, x), v_i\big),$$

where $(b_{-i}, x) = (b_1, \dots, b_{i-1}, x, b_{i+1}, \dots, b_n)$. A Nash equilibrium b is *efficient* if for all $i, j \in \{1, \dots, n\}$, $b_i > b_j$ whenever $v_i > v_j$.

In the (symmetric) *incomplete information* model values v_i are drawn independently from a continuous distribution with density function f, cumulative distribution function F, and support $[0, \bar{v}]$ for some finite $\bar{v} \in \mathbb{R}_+$ we assume to be common knowledge among the bidders.[4] Our results in addition require existence and boundedness of the first three derivatives of F. Since valuations are independent and identically distributed, an efficient assignment for all value profiles can only be obtained from a symmetric profile $(b, \dots, b)$ for some bidding function $b : \mathbb{R} \to \mathbb{R}$. The quantity of interest for strategic considerations under incomplete information is the *expected utility* $u_i^b(x, v_i)$ of bidder i with value v_i given that it bids $x \in \mathbb{R}$ and all other bidders use bidding function b, which is given by

$$u_i^b(x, v_i) = \mathbb{E}_{v_j \sim F, j \neq i}\Big[u_i\Big(v_i, \big(b(v_1), \dots, b(v_{i-1}), x, b(v_{i+1}), \dots, b(v_n)\big)\Big)\Big].$$

Bidding function b then is a *Bayes-Nash equilibrium* if no bidder has an incentive to change its bid, i.e., if for all $i \in \{1, \dots, n\}$ and $v_i \in [0, \bar{v}]$,

$$u_i^b(b(v_i), v_i) = \max_{x \in \mathbb{R}} u_i^b(x, v_i). \tag{1}$$

A Bayes-Nash equilibrium b is *efficient* if it is increasing almost everywhere.

A mechanism that achieves efficiency in both Nash and Bayes-Nash equilibrium is the Vickrey-Clarke-Groves (VCG) mechanism. It uses allocation rule g and a payment rule p^β that charges each bidder its externality on the other bidders, which is equal to the additional utility bidders assigned lower positions would obtain by moving up one position. Denoting by $b_{(i)}$ the $(n-i+1)$st order statistic of b, such that $b_{(1)} \geq \dots \geq b_{(n)}$, and using the convention that $b_{(i)} = 0$ when $i > n$,

$$p_i^\beta(b) = \sum_{j=g_i(b)}^{k} (\beta_j - \beta_{j+1}) b_{(j+1)}.$$

It is well known and not difficult to see that the VCG mechanism makes it optimal for each bidder to bid its true valuation irrespective of the bids of others, which is a stronger

[4] An analytical characterization of equilibria in the case of non-identical distributions is, unfortunately, well beyond the state of the art even for very simple settings. For a single item and two bidders with values drawn uniformly from distinct intervals, for example, this question was posed by Vickrey [36] and answered only recently, almost half a century later, by Kaplan and Zamir [25].

property than those required of a Nash or Bayes-Nash equilibrium. The resulting assignment is efficient. The resulting outcome of assignment and payments is in fact the bidder-optimal core outcome, and we will refer to it by that name.

While computation of payments in the VCG mechanism requires knowledge of the vector β of relative values, we will be interested instead in the ability of mechanisms to support an efficient outcome in equilibrium when only an inaccurate estimate $\alpha \in \mathbb{R}^k_{\geq}$ of β is available to the auctioneer. To this end we consider parameterized variants of the three mechanisms that have been used and studied most extensively: the α-VCG mechanism, the α-GFP mechanism, and the α-GSP mechanism. The three mechanisms all use allocation rule g, and their payment rules p^V, p^F, and p^S respectively charge a bidder its externality, its bid on the position it is assigned, and the next-lower bid on that position. Using the convention that $\alpha_j = 0$ when $j > k$,

$$\begin{aligned}
p_i^V(b) &= \sum_{j=g_i(b)}^{k} (\alpha_j - \alpha_{j+1}) b_{(j+1)}, \\
p_i^F(b) &= \alpha_{g_i(b)} b_i, \quad \text{and} \\
p_i^S(b) &= \alpha_{g_i(b)} b_{(g_i(b)+1)}.
\end{aligned}$$

We will sometimes drop superscripts when the mechanism we are referring to is clear from the context.

3 Complete Information

We begin our analysis with the complete-information case. Here, when $\alpha = \beta$, the α-VCG mechanism has a truthful equilibrium, the α-GSP mechanism has an equilibrium that yields the bidder-optimal core outcome [19, 34], and the α-GFP mechanism may not have any equilibrium [12]. When $\alpha \neq \beta$ the α-VCG mechanism loses its truthfulness, and it makes sense to ask under what conditions the α-VCG mechanism and the α-GSP mechanism possess an efficient equilibrium. To build intuition we first look at the special case with three positions and three bidders, before moving on to the general case. Whereas the special case can be analyzed by comparing the equilibrium conditions of the two mechanisms more or less directly, and thus lends itself to the illustration of the relative strength of these conditions, the analysis of the general case will require an additional insight.

3.1 Three Positions and Three Bidders

In the special case, valuations are given by vectors $v \in \mathbb{R}^3$ and $\beta \in \mathbb{R}^3_{\geq}$ while mechanisms use a vector $\alpha \in \mathbb{R}^3_{\geq}$ that may differ from β. Our goal will be to understand which combinations of α and β allow for the existence of a bid profile $b \in \mathbb{R}^3$ that is an equilibrium and leads to an efficient assignment. Assuming without loss of generality that $v_1 \geq v_2 \geq v_3 > 0$, efficiency requires that

$$b_1 \geq b_2 \geq b_3. \tag{2}$$

For b to be an equilibrium, none of the bidders may benefit from raising or lowering their respective bid and being assigned a different position. For the α-VCG mechanism this means that

$$\beta_1 v_1 - (\alpha_1 - \alpha_2)b_2 - (\alpha_2 - \alpha_3)b_3 \geq \beta_2 v_1 - (\alpha_2 - \alpha_3)b_3, \tag{3}$$
$$\beta_1 v_1 - (\alpha_1 - \alpha_2)b_2 - (\alpha_2 - \alpha_3)b_3 \geq \beta_3 v_1, \tag{4}$$
$$\beta_2 v_2 - (\alpha_2 - \alpha_3)b_3 \geq \beta_1 v_2 - (\alpha_1 - \alpha_2)b_1 - (\alpha_2 - \alpha_3)b_3, \tag{5}$$
$$\beta_2 v_2 - (\alpha_2 - \alpha_3)b_3 \geq \beta_3 v_2, \tag{6}$$
$$\beta_3 v_3 \geq \beta_1 v_3 - (\alpha_1 - \alpha_2)b_1 - (\alpha_2 - \alpha_3)b_2, \tag{7}$$
$$\beta_3 v_3 \geq \beta_2 v_3 - (\alpha_2 - \alpha_3)b_2. \tag{8}$$

There is no upper bound on b_1 and no lower bound on b_3 except $b_3 \geq 0$. Hence, whenever there is a solution, there is one in which we can increase b_1 and set $b_3 = 0$. Setting b_1 to a large value and $b_3 = 0$ satisfies (2), (5), (6), and (7). With this choice of b_3, and since $\beta_2 v_1 \geq \beta_3 v_1$, (4) is implied by (3). The α-VCG mechanism thus possesses an efficient equilibrium if and only if there exists a bid b_2 such that

$$(\alpha_1 - \alpha_2)b_2 \leq (\beta_1 - \beta_2)v_1, \tag{9}$$
$$(\alpha_2 - \alpha_3)b_2 \geq (\beta_2 - \beta_3)v_3. \tag{10}$$

For the α-GSP mechanism the equilibrium conditions require that

$$\beta_1 v_1 - \alpha_1 b_2 \geq \beta_2 v_1 - \alpha_2 b_3, \tag{11}$$
$$\beta_1 v_1 - \alpha_1 b_2 \geq \beta_3 v_1, \tag{12}$$
$$\beta_2 v_2 - \alpha_2 b_3 \geq \beta_1 v_2 - \alpha_1 b_1, \tag{13}$$
$$\beta_2 v_2 - \alpha_2 b_3 \geq \beta_3 v_2, \tag{14}$$
$$\beta_3 v_3 \geq \beta_1 v_3 - \alpha_1 b_1, \tag{15}$$
$$\beta_3 v_3 \geq \beta_2 v_3 - \alpha_2 b_2. \tag{16}$$

There is again no upper bound on b_1, and setting b_1 to a large value satisfies (13) and (15). It is, moreover, not difficult to see that (12) is implied by (11) and (14): by (14), $\alpha_2 b_3 \leq (\beta_2 - \beta_3)v_2$, so (11) implies that $\beta_1 v_1 - \alpha_1 b_2 \geq \beta_2 v_1 - (\beta_2 - \beta_3)v_2$; since $v_1 \geq v_2$, this in turn implies (12). The α-GSP mechanism thus possesses an efficient equilibrium if and only if there exist bids $b_2 \geq b_3$ such that

$$\alpha_1 b_2 \leq (\beta_1 - \beta_2)v_1 + \alpha_2 b_3, \tag{17}$$
$$\alpha_2 b_3 \leq (\beta_2 - \beta_3)v_2, \tag{18}$$
$$\alpha_2 b_2 \geq (\beta_2 - \beta_3)v_3. \tag{19}$$

To see that the constraints for the α-GSP mechanism are generally weaker than those for the α-VCG mechanism, note that the former can be satisfied even under the additional restriction that $b_2 = b_3$ if there exists a bid b_2 such that

$$(\alpha_1 - \alpha_2)b_2 \leq (\beta_1 - \beta_2)v_1, \tag{20}$$

$$\alpha_2 b_2 \geq (\beta_2 - \beta_3)v_3. \tag{21}$$

Indeed, any such bid satisfies (17) and (19), while the smallest such bid satisfies (18) as well. The claim now follows because (20) is identical to (9) and (21) easier to satisfy than (10). The latter comparison will in fact be strict when $\alpha_3 > 0$.

When $\alpha_3 = 0$ there is no strict separation and the two mechanisms are in fact identical, but this is a viable design choice only in the absence of a fourth bidder, when the payment for the last position is always zero. When there is a fourth bidder, then for both mechanisms $\alpha_3 > 0$ becomes a necessary condition for the existence of an efficient equilibrium and the separation between the mechanisms is strict. A formal treatment of the case with three positions and four bidders is given in Appendix A. This treatment also suggests that the analysis becomes significantly more difficult as the number of positions and bidders increases and can no longer be solved by a straightforward comparison of the respective equilibrium conditions.

To further illustrate the separation between the two mechanisms, note that when $\beta_i \neq \beta_j$ and $\alpha_i \neq \alpha_j$ for all $i \neq j$, (9) and (10) can be satisfied if and only if

$$\alpha_2 \geq \frac{\alpha_1(\beta_2 - \beta_3)v_3 + \alpha_3(\beta_1 - \beta_2)v_1}{(\beta_1 - \beta_2)v_1 + (\beta_2 - \beta_3)v_3},$$

and (17), (18), and (19) if and only if

$$\alpha_2 \geq \frac{\alpha_1(\beta_2 - \beta_3)v_3}{(\beta_1 - \beta_2)v_1 + (\beta_2 - \beta_3)v_3}.$$

We compare these bounds in Figure 2. The figure suggests that only an underestimation of β is problematic, while efficient equilibria are preserved by both mechanisms when $\alpha \geq \beta$. The analysis in Appendix A shows that this, also, is an artifact of the case with three positions and three bidders and ceases to hold when there is an additional bidder.

Intuition for the reasons underlying the separation can be gained by considering the upper and lower bounds on the bid b_2 of the second bidder in an efficient equilibrium when β_2 is fixed and α_2 varies. Figure 3 shows an illustration for parameters on the dotted line of Figure 2. Bid b_2 is subject to an upper bound because it contributes to the payment for the first position and setting it too high would mean that the first bidder would prefer the second position to the first. The upper bound is imposed by (3) for the α-VCG mechanism and by (11) for the α-GSP mechanism. While the former constraint depends on b_3, this dependence affects the first and second position in the same way and cancels out. The upper bound is thus the same for both mechanism, and is shown in gray in Figure 3. A lower bound applies to b_2 because it determines the hypothetical payment of the third bidder if that bidder were to bid above b_2 and thus be assigned the second position. The lower bound is imposed by (8) for the α-VCG mechanism and by (16) for the α-GSP mechanism. The

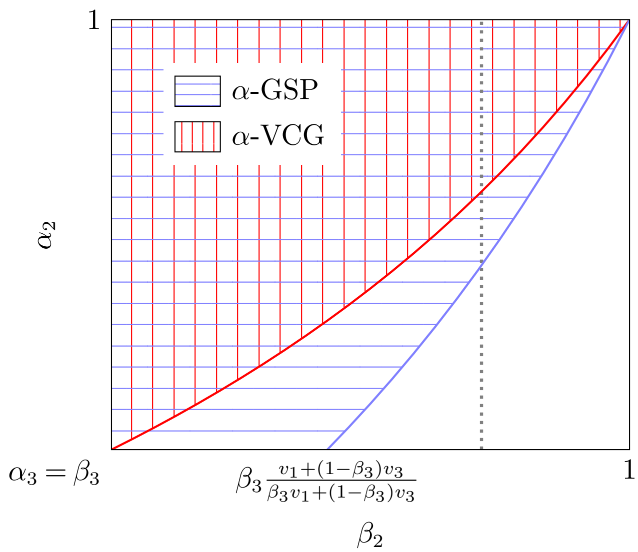

Figure 2: Comparison of the α-GSP and α-VCG mechanisms under complete information, for a setting with three positions and three bidders where $\beta_1 = \alpha_1 = 1$ and $0 < \beta_3 = \alpha_3 < 1$. The hatched areas indicate the combinations of α_2 and β_2 for which the two mechanisms respectively possess an efficient equilibrium. The dotted line illustrates the performance of the mechanisms for a particular value of β_2. When $v_1 = 10$, $v_3 = 6$, and $\alpha_3 = \beta_3 = 0.3$, this line would lie at $\beta_2 = 0.8$ and would intersect the curve for the α-GSP mechanism at $\alpha_2 = 0.6$ and that for the α-VCG mechanism at $\alpha_2 = 0.72$. Any point on the dotted line between these two intersection points corresponds to a value of α_2 for which the α-GSP mechanism possesses an efficient equilibrium and the α-VCG mechanism does not.

hypothetical payment is higher in the α-GSP mechanism and prevents the third bidder from bidding above b_2 also for smaller values of b_2. As a consequence, the α-GSP mechanism still possesses an efficient equilibrium for smaller values of α_2. The same general intuition applies in settings with arbitrary numbers of bidders and positions when they are viewed from the perspective of a particular bidder, but the interactions among the different bids quickly make it impractical to establish equilibrium existence by comparing the equilibrium conditions directly.

3.2 The General Case

We proceed to show that superiority of the α-GSP mechanism over the α-VCG mechanism in preserving efficient equilibria holds in general. The following result establishes a weak superiority for arbitrary numbers of bidders and positions and arbitrary valuations. Examples in which only the α-GSP mechanism preserves an efficient equilibrium are straightforward to construct, and indeed we have already done so for a specific setting.

Theorem 1. *Let $\alpha, \beta \in \mathbb{R}^k_{\geq}$, $v \in \mathbb{R}^n$. Then the α-GSP mechanism possesses an efficient Nash equilibrium for valuations given by β and v whenever the α-VCG mechanism does.*

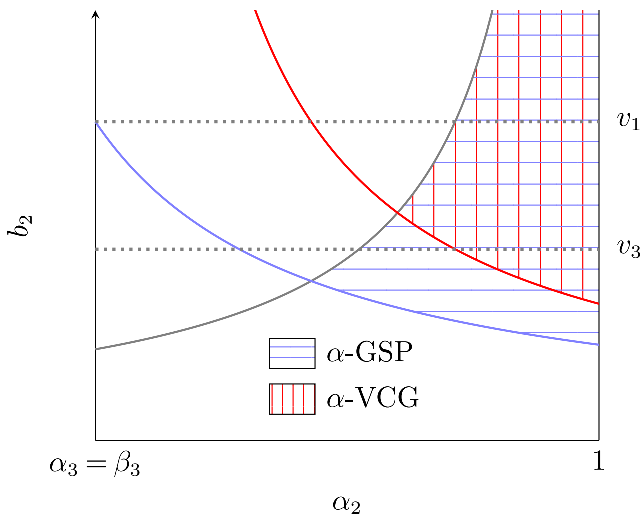

Figure 3: Comparison of the α-GSP and α-VCG mechanisms for parameters on the dotted line of Figure 2. The hatched areas indicate, for each of the two mechanisms, possible bids b_2 of the second bidder in an efficient equilibrium. When $v_1 = 10$, $v_3 = 6$, $\alpha_3 = \beta_3 = 0.3$, and $\beta_2 = 0.8$, the upper bound intersects the lower bound for the α-GSP mechanism at $\alpha_2 = 0.6$ and the lower bound for the α-VCG mechanism at $\alpha_2 = 0.72$. The dotted lines indicate the values of v_1 and v_3 and thus the possible range of values of v_2.

Rather than by comparing the equilibrium conditions of the two mechanisms directly, as we have done in the special case, we prove the theorem by appealing to the stronger requirement of envy-freeness, which has played a significant role also in earlier work on VCG and GSP position auctions [34, 19]. We will see that in the α-VCG mechanism existence of an efficient equilibrium implies existence of an efficient envy-free equilibrium, which in turn implies existence of an efficient envy-free equilibrium, and thus of an efficient equilibrium, in the α-GSP mechanism.

The first implication, from existence of an efficient Nash equilibrium in the α-VCG mechanism to existence of an efficient bid profile satisfying envy-freeness, is shown in Lemma 1. Here, bid profile b is called envy-free if no bidder prefers a different position to the one it is currently assigned at the *current* payment for the former, i.e., if for all $i \in \{1, \dots, n\}$,

$$\beta_{g_i(b)} v_i - p_i(b) = \max_{j \in \{1,\dots,n\}} \beta_{g_j(b)} v_i - p_j(b). \tag{22}$$

For both the α-VCG mechanism and the α-GSP mechanism, envy-freeness implies the equilibrium condition because the current payment is a lower bound on the actual payment of the bidder if by either mechanism it was assigned that position. Moreover, and in contrast to the equilibrium condition, envy-freeness can be viewed as a requirement that depends only on the allocation and payments and not on the underlying mechanism. We can thus complete the proof by establishing the existence of a mapping from bid profiles in the α-VCG mechanism to bid profiles in α-GSP mechanism that preserves allocation and payments, which we do in Lemma 2.

Assume without loss of generality that $v_1 \geq v_2 \geq \cdots \geq v_n$ and that in an efficient allocation, for $1 \leq i \leq \min\{n, k\}$, bidder i is assigned position i. Specializing and rearranging (1), a bid profile b with $b_1 \geq \cdots \geq b_n$ is a Nash equilibrium of the α-VCG mechanism if for all $i, j \in \{1, \ldots, n\}$,

$$(\alpha_j - \alpha_{j+1})b_j \geq (\beta_j - \beta_i)v_i - \sum_{t=j+1}^{i-1}(\alpha_t - \alpha_{t+1})b_t \qquad \text{if } j < i, \tag{23}$$

$$(\alpha_i - \alpha_{i+1})b_{i+1} \leq (\beta_i - \beta_j)v_i - \sum_{t=i+1}^{j-1}(\alpha_t - \alpha_{t+1})b_{t+1} \qquad \text{if } j > i. \tag{24}$$

These conditions constrain the utility of bidder i if instead of position i it was assigned a position j that is respectively above or below i. Note that in the latter case the payment of bidder i for position j is equal to the current payment for this position, where in the former case it may be higher. Specializing (22), a bid profile b with $b_1 \geq \cdots \geq b_n$ is envy-free if for all $i, j \in \{1, \ldots, n\}$, in addition to (24) and instead of (23),[5]

$$(\alpha_j - \alpha_{j+1})b_{j+1} \geq (\beta_j - \beta_i)v_i - \sum_{t=j+1}^{i-1}(\alpha_t - \alpha_{t+1})b_{t+1} \qquad \text{if } j < i. \tag{25}$$

Envy-freeness is a stronger requirement than that of being an equilibrium, but we will see that it comes for free in the sense that existence of an efficient equilibrium automatically implies existence of an efficient equilibrium satisfying envy-freeness.

Lemma 1. *Let $\alpha, \beta \in \mathbb{R}^k_{\geq}$ and $v \in \mathbb{R}^n_{\geq}$, and assume that the α-VCG mechanism possesses an efficient equilibrium. Then the α-VCG mechanism possesses an efficient equilibrium that is envy-free.*

Proof. We will show existence and envy-freeness of a particular type of efficient equilibrium that we will call bidder-pessimal, in which each of the bids $b_2, \ldots, b_n$ is maximal among all efficient equilibria.[6]

First note that (23) only imposes lower bounds on the bids and remains satisfied when bids are increased. For (24), the case where $j = i + 1$ implies all other cases, because

$$\begin{aligned}(\beta_i - \beta_j)v_i - \sum_{t=i+1}^{j-1}(\alpha_t - \alpha_{t+1})b_{t+1} &\geq (\beta_i - \beta_j)v_i - \sum_{t=i+1}^{j-1}(\beta_t - \beta_{t+1})v_t \\ &\geq (\beta_i - \beta_j)v_i - \sum_{t=i+1}^{j-1}(\beta_t - \beta_{t+1})v_i \\ &= (\beta_i - \beta_{i+1})v_i,\end{aligned}$$

[5] Condition (25) is, unlike (23), symmetric to (24). Varian [34] therefore refers to bid profiles satisfying (25) and (24) as *symmetric equilibria*.

[6] In the case where $\alpha = \beta$, the truthful equilibrium of the α-VCG mechanism is bidder-*optimal* among all envy-free outcomes, i.e., its bids and payments are minimal [26].

where the first inequality holds because, by the fact that b is an efficient equilibrium and hence by (24), $(\alpha_t - \alpha_{t+1})b_{t+1} \leq (\beta_t - \beta_{t+1})v_t$ when $i+1 \leq t \leq j-1$, and the second inequality because, by efficiency, $v_t \leq v_i$ for all such t. Bid b_i, for $i \in \{2, \ldots, n\}$, thus is subject to only two upper bounds, $b_i \leq b_{i-1}$ by efficiency and $(\alpha_{i-1} - \alpha_i)b_i \leq (\beta_{i-1} - \beta_i)v_{i-1}$ by (24). Increasing each of these bids as much as possible yields a bid profile b such for all $i \in \{2, \ldots, n\}$,

$$b_i = \begin{cases} \min\left(b_{i-1}, \frac{(\beta_{i-1}-\beta_i)v_{i-1}}{\alpha_{i-1}-\alpha_i}\right) & \text{if } \alpha_{i-1} \neq \alpha_i, \\ b_{i-1} & \text{otherwise.} \end{cases} \tag{26}$$

We now claim that b satisfies (25) and begin by showing this for the special case where $j = i-1$, which requires that for all $i \in \{2, \ldots, n\}$,

$$(\alpha_{i-1} - \alpha_i)b_i \geq (\beta_{i-1} - \beta_i)v_i. \tag{27}$$

By (26) it suffices to distinguish two cases. If $\alpha_{i-1} \neq \alpha_i$ and $b_i = (\beta_{i-1} - \beta_i)v_{i-1}/(\alpha_{i-1} - \alpha_i)$, then

$$(\alpha_{i-1} - \alpha_i)b_i = (\beta_{i-1} - \beta_i)v_{i-1} \geq (\beta_{i-1} - \beta_i)v_i,$$

where the inequality holds because $v_i \geq v_{i+1}$. If instead $b_i = b_{i-1}$, then

$$(\alpha_{i-1} - \alpha_i)b_i = (\alpha_{i-1} - \alpha_i)b_{i-1} \geq (\beta_{i-1} - \beta_i)v_i,$$

where the inequality holds by (23).

For the general case let $i, j \in \{1, \ldots, n\}$ with $j < i$. Then

$$\begin{aligned}
(\beta_j - \beta_i)v_i - \sum_{t=j+1}^{i-1}(\alpha_t - \alpha_{t+1})b_{t+1} &\leq (\beta_j - \beta_i)v_i - \sum_{t=j+1}^{i-1}(\beta_t - \beta_{t+1})v_{t+1} \\
&\leq (\beta_j - \beta_i)v_i - \sum_{t=j+1}^{i-1}(\beta_t - \beta_{t+1})v_i \\
&= (\beta_j - \beta_{j+1})v_i \\
&\leq (\beta_j - \beta_{j+1})v_{j+1} \\
&\leq (\alpha_j - \alpha_{j+1})b_{j+1},
\end{aligned}$$

where the first and last inequality hold because, by (27), $(\alpha_t - \alpha_{t+1})b_{t+1} \geq (\beta_t - \beta_{t+1})v_{t+1}$ for $t = j+1, \ldots, i-1$ and $(\beta_j - \beta_{j+1})v_{j+1} \leq (\alpha_j - \alpha_{j+1})b_{j+1}$, and the second and third inequality because $v_{t+1} \geq v_i$ when $t+1 \leq i$ and $v_i \leq v_j$ when $j < i$.[7] □

[7] In extending the claim from the special to the general case we have in fact shown that, subject to efficiency, *local* envy-freeness with regard to the position directly above implies envy-freeness with regard to all higher positions. Similar results have appeared in prior work [e.g., 34, Fact 5], but only for the case where $\alpha = \beta$.

We proceed to show that any bid profile in the α-VCG mechanism can be mapped to a bid profile that in the α-GSP mechanism yields the same allocation and payments. Applying this mapping to an efficient envy-free bid profile like the one identified by Lemma 1, and noting that envy-freeness implies the equilibrium condition, then shows Theorem 1.

Lemma 2. *Let $\alpha, \beta \in \mathbb{R}^k_{\geq}$ and $v, b \in \mathbb{R}^n_{\geq}$. Let $b^S \in \mathbb{R}^n$ such that for all $i \in \{1, \ldots, n\}$,*

$$b_i^S = \begin{cases} b_2^S & \text{if } i = 1, \\ \frac{p_{i-1}^V(b)}{\alpha_{i-1}} & \text{if } i \in \{2, \ldots, k+1\} \text{ and } \alpha_{i-1} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $b_1^S \geq \cdots \geq b_n^S$ and for all $i \in \{1, \ldots, n\}$, $p_i^S(b^S) = p_i^V(b)$.

Proof. Note that $b_1^S = b_2^S$. Let $j = \min\{i : \alpha_i = 0\}$ and note that $b_i^S = 0$ when $i > j$. For the first part of the claim it thus suffices to show that $b_i^S \geq b_{i+1}^S$ for $i = 2, \ldots, j-1$, which we do in two steps. First, for all $i \in \{2, \ldots, j\}$,

$$b_i^S = \frac{p_{i-1}^V}{\alpha_{i-1}} = \frac{\sum_{t=i-1}^k (\alpha_t - \alpha_{t+1}) b_{t+1}}{\alpha_{i-1}} \leq \frac{\sum_{t=i-1}^k (\alpha_t - \alpha_{t+1}) b_i}{\alpha_{i-1}} = \frac{(\alpha_{i-1} - \alpha_{k+1}) b_i}{\alpha_{i-1}} = b_i, \quad (28)$$

where the first two equalities respectively hold by definition of b_i^S and p_{i-1}^V, the inequality because $b_1 \geq \cdots \geq b_n$, and the last equality because, by convention, $\alpha_{k+1} = 0$. Then, for all $i \in \{2, \ldots, j-1\}$,

$$\begin{aligned} b_i^S = \frac{p_{i-1}^V}{\alpha_{i-1}} &= \frac{(\alpha_{i-1} - \alpha_i) b_i + p_i^V}{\alpha_{i-1}} \\ &= \frac{(\alpha_{i-1} - \alpha_i) b_i + \alpha_i b_{i+1}^S}{\alpha_{i-1}} \geq \frac{(\alpha_{i-1} - \alpha_i) b_{i+1}^S + \alpha_i b_{i+1}^S}{\alpha_{i-1}} = b_{i+1}^S, \end{aligned}$$

where the first and third equalities hold by definition of b_i^S, the second equality exploits the recursive nature of the definition of p_{i-1}^V, and the inequality uses that $b_i \geq b_{i+1}$ and that, by (28), $b_{i+1} \geq b_{i+1}^S$.

The second part of the claim is satisfied for $i < j$ because $p_i^S = \alpha_i b_{i+1}^S = \alpha_i p_i^V / \alpha_i = p_i^V$, and for $i \geq j$ because $\alpha_i = 0$ for all $i \geq j$ and thus $p_i^S = p_i^V = 0$. □

The above analysis in fact shows that any envy-free equilibrium of the α-VCG mechanism is preserved by the α-GSP mechanism. Since the bidder-optimal core outcome is envy-free [26], we thus have the following.

Corollary 1. *Let $\alpha, \beta \in \mathbb{R}^k_{\geq}$, $v \in \mathbb{R}^n$. Then the α-GSP mechanism obtains the bidder-optimal core outcome in a Nash equilibrium for valuations given by β and v whenever the α-VCG mechanism does.*

4 Incomplete Information

We now turn to incomplete-information environments, where bidders only possess probabilistic information regarding one another's valuations. When $\alpha = \beta$, the α-VCG mechanism of course maintains its truthful dominant-strategy equilibrium. The α-GSP mechanism may fail to possess an efficient equilibrium even when $\alpha = \beta$ and the bidders have identically distributed valuations.[8] When $\alpha = \beta$ and valuations are identically distributed then α-GFP possesses a unique Bayes-Nash equilibrium, and this equilibrium is efficient [12].

Given these results it is natural to ask how successful the α-VCG and α-GFP mechanisms are in maintaining an efficient equilibrium outcome when $\alpha \neq \beta$ and valuations are distributed identically. We show that, perhaps surprisingly, the non-truthful mechanism is again more robust, for arbitrary values of α and β and independent and identically distributed valuations according to any distribution satisfying mild technical conditions. Our analysis uses Myerson's classical characterization of possible equilibrium bids to identify, for either of the two mechanisms, conditions on α and β that are necessary and sufficient for equilibrium existence. The conditions for the α-VCG mechanism turn out to be more demanding. Just as we did for complete-information environments, we begin by considering a special case, this time with two positions, three bidders, and valuations drawn uniformly at random from the unit interval. The special case is used to build intuition, and introduce the necessary machinery, for the general result.

It is known that the first-price auction may not possess an efficient equilibrium in certain asymmetric settings [28]. Together with our result this shows that in the general case, when valuations are not necessarily identically distributed, neither mechanism dominates the other.

4.1 Two Positions and Three Bidders

Let v_1, v_2, v_3 be drawn independently from the uniform distribution on $[0, 1]$. Let $\alpha, \beta \in \mathbb{R}^2_{\geq}$ with $\alpha_2, \beta_2 > 0$, and assume without loss of generality that $\alpha_1 = \beta_1 = 1$. Our goal will again be to characterize the values of α and β for which given mechanisms of interest, in this case the α-GFP and α-VCG mechanisms, admit an efficient equilibrium. Behavior under incomplete information can be described by a profile of bidding functions, one for each bidder, that map the bidder's value to its bid. It is clear that in a symmetric setting like ours efficient outcomes can only result from symmetric bidding functions, so we will be interested in functions $b^F : \mathbb{R} \to \mathbb{R}$ that yield an efficient equilibrium in the α-GFP mechanism and functions $b^V : \mathbb{R} \to \mathbb{R}$ that achieve the same in the α-VCG mechanism.

The standard technique for equilibrium analysis under incomplete information uses a seminal result of Myerson that characterizes the expected allocation and payments in equilibrium in terms of the allocation probabilities induced by a mechanism and bidders' bidding

[8] Gomes and Sweeney [20] gave a characterization of those values of α that enable equilibrium existence in this case. The result can be strengthened in our setting to show that for some values of β no choice of α leads to an efficient equilibrium.

functions. The result was originally formulated for truthful mechanisms, but equivalent conditions exist for arbitrary bidding functions that instead of being in equilibrium provide a best response among values in their range. The latter is obviously a necessary condition for equilibrium, and can be turned into a sufficient condition by arguing that no better response exists outside the range. For our setting and notation we have the following result.

Lemma 3 (Myerson [32]). *Consider a position auction for an environment with n bidders, k positions, and $\beta \in \mathbb{R}^k_{\geq}$. Assume that bidders use a bidding function b with range X, and that a bidder with value v is consequently assigned position $s \in \{1, \dots, k\}$ with probability $P_s(v)$. Then $u(b(v), v) = \max_{x \in X} u(x, v)$ for all $v \in [0, \bar{v}]$ if and only if the following holds:*

(a) the expected allocation $\sum_{s=1}^{k} P_s(v)\beta_s$ is non-decreasing in v, and

(b) the payment function p satisfies

$$\mathbb{E}[p(v)] = p(0) + \sum_{s=1}^{k} \beta_s \int_0^v \frac{dP_s(z)}{dz} z \, dz. \tag{29}$$

All mechanisms we consider set $p(0) = 0$ and use an efficient allocation rule, for which

$$P_s(v) = \binom{n-1}{s-1} (1 - F(v))^{s-1} (F(v))^{n-s}$$

and (a) is satisfied. Together with our assumptions on F, efficiency mandates further that b must increase almost everywhere.

In the special case with two positions and three bidders with values distributed uniformly on the unit interval we have that $P_1(v) = F^2(v) = v^2$ and $P_2(v) = \binom{2}{1} F(v)(1-F(v)) = 2v(1-v)$, payments in any efficient equilibrium can thus be described by a function $p^E : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{aligned} \mathbb{E}[p^E(v)] &= \beta_1 \int_0^v \frac{dP_1(z)}{dz} z \, dz + \beta_2 \int_0^v \frac{dP_2(z)}{dz} z \, dz \\ &= \frac{2}{3} \beta_1 v^3 + \beta_2 v^2 - \frac{4}{3} \beta_2 v^3. \end{aligned} \tag{30}$$

A candidate equilibrium bidding function for the α-GFP mechanism can now be obtained by writing the expected payment in terms of bidding function b^F, equating the resulting expression with (30), and solving for b^F. In the α-GFP mechanism a bidder with value v that is allocated position s pays $\alpha_s b^F(v)$, its expected payment therefore satisfies

$$\begin{aligned} \mathbb{E}[p^F(v)] &= P_1(v)\alpha_1 b^F(v) + P_2(v)\alpha_2 b^F(v) \\ &= (\alpha_1 v^2 + 2\alpha_2 v - 2\alpha_2 v^2) b^F(v). \end{aligned} \tag{31}$$

By Lemma 3 the expressions in (30) and (31) must be the same. Equating them yields

$$b^F(v) = \frac{2/3 \cdot v^3 - 4/3 \cdot \beta_2 v^3 + \beta_2 v^2}{v^2 - 2\alpha_2 v^2 + 2\alpha_2 v}$$

when $v > 0$, and we can set $b^F(0) = 0$ for convenience.[9] Bidding below $b^F(0) = 0$ is impossible, bidding above $b^F(\bar{v})$ is dominated,[10] and b^F satisfies the second condition of Lemma 3 by construction. The α-GFP mechanism thus has an efficient equilibrium if and only if b^F is increasing almost everywhere. Taking the derivative we obtain

$$\frac{db^F(v)}{dv} = \frac{(\frac{4}{3}v - \frac{8}{3}\beta_2 v + \beta_2)(v - 2\alpha_2 v + 2\alpha_2)}{(v - 2\alpha_2 v + 2\alpha_2)^2} - \frac{(1-2\alpha_2)(\frac{2}{3}v^2 - \frac{4}{3}\beta_2 v^2 + \beta_2 v)}{(v - 2\alpha_2 v + 2\alpha_2)^2}.$$

The sign of this expression is determined by the sign of its numerator, and it turns out that the numerator is positive at 0 and, depending on the value of β_2, either non-decreasing everywhere on $[0, 1]$ or decreasing everywhere on $[0, 1]$. Indeed, $db^F(v)/dv|_{v=0} = \beta_2/(2\alpha_2) > 0$, and the derivative of the numerator, $(4/3 - 8/3\beta_2)(v - 2\alpha_2 v + 2\alpha_2)$, is non-negative when $\beta_2 \leq 1/2$ and negative when $\beta_2 > 1/2$. In the case where $\beta_2 > 1/2$ we need that

$$\left.\frac{db^F(v)}{dv}\right|_{v=1} = \left(\frac{4}{3} - \frac{5}{3}\beta_2\right) - (1-2\alpha_2)\left(\frac{2}{3} - \frac{1}{3}\beta_2\right) \geq 0,$$

which holds when

$$\alpha_2 \geq \frac{2\beta_2 - 1}{2 - \beta_2}.$$

We conclude that the α-GFP mechanism possesses an efficient equilibrium if and only if $\beta_2 \leq 1/2$ or $\alpha_2 \geq (2\beta_2 - 1)/(2 - \beta_2)$.

Analogously, in the α-VCG mechanism, the payment of a bidder with value v satisfies

$$\begin{aligned}\mathbb{E}[p^V(v)] &= P_1(v)\left[(\alpha_1 - \alpha_2)\int_0^v \frac{2t}{v^2} b^V(t)\,dt + \alpha_2 \int_0^v \frac{2(v-t)}{v^2} b^V(t)\,dt\right] + P_2(v)\alpha_2 \int_0^v \frac{1}{v} b^V(t)\,dt \\ &= (2\alpha_1 - 4\alpha_2)\int_0^v t b^V(t)\,dt + 2\alpha_2 \int_0^v b^V(t)\,dt, \qquad (32)\end{aligned}$$

where $2t/v^2 = 2F(t)f(t)/F(v)^2$ and $2(v-t)/v^2 = 2F(v-t)f(t)/F(v)^2$ are the densities of the second and third highest values given that the bidder's value v is the highest, and $1/v = f(t)/F(v)$ is the density of the third highest value given that v is the second highest. By Lemma 3 the expressions in (30) and (32) must again be the same. Taking the derivatives of both and solving for $b^V(v)$ yields

$$b^V(v) = \frac{2v^2 - 4\beta_2 v^2 + 2\beta_2 v}{2v - 4\alpha_2 v + 2\alpha_2}$$

when $v < 1$, and we can extend b^V appropriately when $v = 1$.[11] By the same argument as before, the α-VCG mechanism has an efficient equilibrium if and only if b^V is increasing

[9] Application of l'Hospital's rule shows that $\lim_{v\to 0} b^F(v) = 0$, so this choice makes b^F increasing.

[10] Since equilibrium bidding functions must be increasing almost everywhere, bidding above $b^F(\bar{v})$ would not increase the probability of winning, and it would also not lead to a lower payment.

[11] We have assumed that $\alpha_2 > 0$, so the denominator vanishes only when $v = \alpha_2 = 1$. If $\beta_2 < 1$, then $\lim_{v\to 1} b^V(v) = \infty$. If $\beta_2 = 1$, application of l'Hospital's rule shows that $\lim_{v\to 1} b^V(v) = 1$.

almost everywhere. Taking the derivative we obtain

$$\frac{db^V(v)}{dv} = \frac{(4v - 8\beta_2 v + 2\beta_2)(2v - 4\alpha_2 v + 2\alpha_2)}{(2v - 4\alpha_2 v + 2\alpha_2)^2} - \frac{(2 - 4\alpha_2)(2v^2 - 4\beta_2 v^2 + 2\beta_2 v)}{(2v - 4\alpha_2 v + 2\alpha_2)^2}.$$

When $\alpha_2 < 1$ the sign of this expression is determined by its numerator, which is positive at 0 and, depending on the value of β_2, either non-decreasing everywhere on $[0, 1]$ or decreasing everywhere on $[0, 1]$. Indeed, $db^F(v)/dv|_{v=0} = \beta_2/\alpha_2 > 0$, and the derivative of the numerator, $(4 - 8\beta_2)(2v - 4\alpha_2 v + 2\alpha_2)$, is non-negative when $\beta_2 \leq 1/2$ and negative when $\beta_2 > 1/2$. When $\beta_2 > 1/2$ we need that

$$\left.\frac{db^V(v)}{dv}\right|_{v=1} = \frac{(4 - 6\beta_2)(2 - 2\alpha_2) - (2 - 4\alpha_2)(2 - 2\beta_2)}{(2 - 2\alpha_2)^2} \geq 0,$$

which for $\alpha_2 < 1$ holds when

$$\alpha_2 \geq 2 - \frac{1}{\beta_2}.$$

When $\alpha_2 = 1$ the above reasoning still applies as long as $v < 1$, so $b^V(v)$ is increasing almost everywhere when

$$\lim_{v \to 1} \frac{db^V(v)}{dv} \geq 0.$$

This is indeed the case, as $\lim_{v\to 1} db^V(v)/dv = \infty$ when $\beta_2 < 1$, and $\lim_{v\to 1} db^V(v)/dv = 1$ when $\beta_2 = 1$ by applying l'Hospital's rule twice. We conclude that the α-VCG mechanism possesses an efficient equilibrium if and only if $\beta_2 \leq 1/2$ or $\alpha_2 \geq 2 - 1/\beta_2$.

It is now not hard to see that the equilibrium condition for the α-GFP mechanism is easier to satisfy than that for the α-VCG mechanism. In fact, for the α-VCG mechanism, efficient equilibria may cease to exist even when α_2 is very close to β_2. When $\beta_2 = 0.8$, for example, any value of $\alpha_2 \geq 0.5$ would suffice for the α-GFP mechanism, while the α-VCG mechanism would require that $\alpha_2 \geq 0.75$. An illustration is provided in Figure 4.

Analogously to the complete-information case, intuition can be gained by considering candidate equilibrium bids for the two mechanisms when β_2 is fixed and α_2 varies. Figure 5 shows an illustration for parameters on the dotted line of Figure 4. The bidding function for the α-GFP mechanism satisfies $b^F(0) = 0$ and $b^F(1) = 2/5$ and is concave for all values of α_2. The bidding function for the α-VCG mechanism satisfies $b^V(1/2) = 1/2$, and is convex when $\alpha_2 \geq \beta_2$ and concave when $\alpha_2 \leq \beta_2$. If α_2 is decreased, bids in the α-GFP mechanism increase for all values, whereas bids in the α-VCG mechanism increase for values below 1/2 and decrease for values above 1/2. To explain this behavior, we recall that by Myerson's characterization expected payments in equilibrium have to remain the same as α_2 changes. In the α-GFP mechanism, where the payment of each bidder is the product of that bidder's bid and an appropriate entry of α or zero, a decrease in α_2 must be compensated through higher bids by bidders likely to be assigned the second position. In the α-VCG mechanism, where payments depend on the bids of bidders assigned lower positions and these bids are weighted by gaps between entries of α, decreasing α_2 simultaneously increases

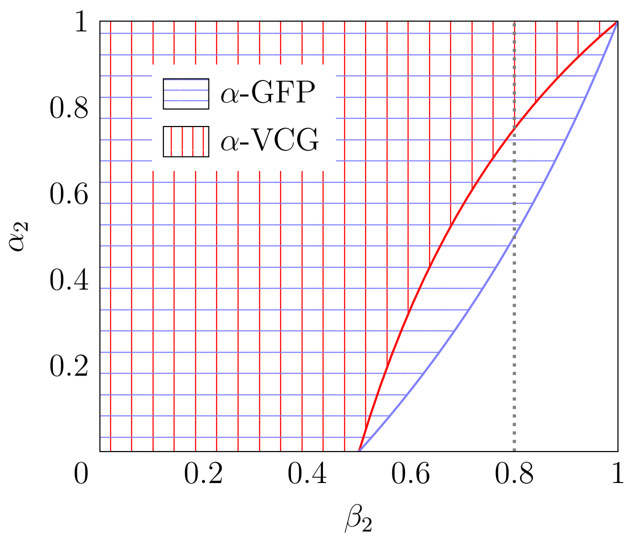

Figure 4: Comparison of the α-GFP and α-VCG mechanisms under incomplete information, for a setting with two positions, three bidders, and valuations drawn independently and uniformly from $[0, 1]$. The hatched areas indicate the combinations of α_2 and β_2 for which the mechanisms respectively possess an efficient equilibrium, when $\alpha_1 = \beta_1 = 1$. The dotted line at $\beta_2 = 0.8$ intersects the curve for the α-GFP mechanism at $\alpha_2 = 0.5$ and that for the α-VCG mechanism at $\alpha_2 = 0.75$. For all points between the intersection points the α-GFP mechanism has an efficient equilibrium and the α-VCG mechanism does not.

the gap between α_1 and α_2 and decreases the gap between α_2 and zero. Holding everything else fixed, this would increase the payment for the first position and decrease the payment for the second position. To counter this, bidders with higher value, who are more likely to receive the second position, have to bid less aggressively, while those with lower value have to bid more aggressively. For both mechanisms the change in bidding behavior ultimately breaks monotonicity of the bidding function and thus existence of an efficient equilibrium. However, the effect in the α-GFP mechanism turns out to be weaker than the combination of effects in the α-VCG mechanism, and equilibrium existence is preserved for smaller values of α_2 in the former.

4.2 The General Case with Identical Distributions

We proceed to establish a weak superiority for any number of positions and bidders and arbitrary symmetric valuation distributions, and note that examples showing a strict separation are straightforward to construct and have indeed been given for a specific setting.

Theorem 2. *Let $\alpha, \beta \in \mathbb{R}^k_{\geq}$. Let $v \in \mathbb{R}^n$, with components drawn independently from a continuous distribution with bounded support. Then the α-GFP mechanism possesses an efficient Bayes-Nash equilibrium for valuations given by β and v whenever the α-VCG mechanism does.*

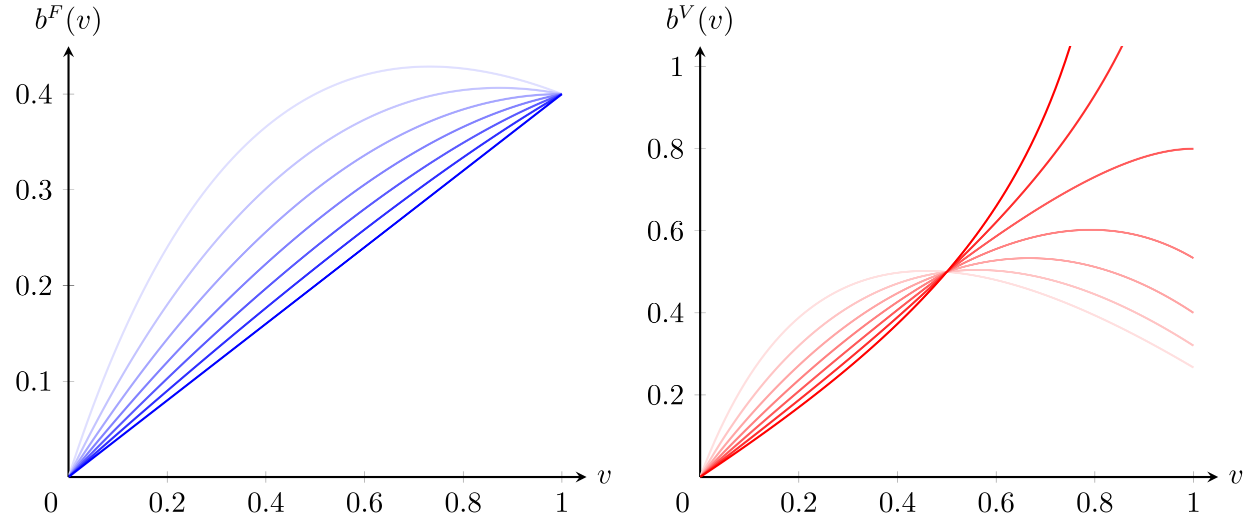

Figure 5: Candidate bidding functions for the α-GFP and α-VCG mechanisms in a setting with three bidders with values distributed uniformly on $[0, 1]$ and two positions with $\alpha_1 = \beta_1 = 1$, $\beta_2 = 0.8$, and $\alpha_2 \in \{0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1\}$ shaded from light to dark. The function for the α-GFP mechanism is increasing almost everywhere when $\alpha_2 \geq 0.5$, the function for the α-VCG mechanism when $\alpha_2 \geq 0.75$.

To obtain this general result we will follow the same basic strategy as in the special case, but will have to overcome two major difficulties on the way.

The first difficulty concerns the equilibrium bidding function for the α-VCG mechanism. Whereas deriving a bidding function for the α-GFP mechanism remains relatively straightforward even for an arbitrary number of positions and arbitrary valuation distributions, the situation becomes significantly more complex for the α-VCG mechanism due to the dependence of its payment rule on the bids for all lower positions. Specifically, when equating the two expressions for the expected payment in equilibrium, (30) and (32) in the special case, and taking derivatives on both sides, the integrand in the latter no longer depends only on t, the variable of integration. Instead, the conditional densities of the values of bidders assigned lower positions introduce a dependence on v. When taking the derivative one would expect to obtain a differential equation, and a closed form solution to this differential equation would be required to continue with the rest of the argument. We take a different route and use a rather surprising combinatorial equivalence to obtain an alternative expression for the expected payment that only depends on t.

A second difficulty arises when trying to show that b^F is increasing for a wider range of values of α and β than b^V. In the special case we could argue directly about the derivatives of the bidding functions, but this type of argument becomes infeasible rather quickly when increasing the number of positions or moving to general value distributions. The key insight that will allow us to generalize the result is that there exist functions $A : \mathbb{R} \to \mathbb{R}$ and $B : \mathbb{R} \to \mathbb{R}$ such that $b^F(v) = A(v)/B(v)$ and $b^V(v) = A'(v)/B'(v)$, where A' and B' respectively denote the derivatives of A and B. This relationship is easily verified for (30) and (32) but continues to hold in general. We use it to show that at the minimum value of v for which $db^F(v)/dv$ is non-positive, should such a value exist, $db^V(v)/dv$ is non-positive as well.

Candidate equilibrium bidding functions We begin by deriving candidate equilibrium bidding functions for the two mechanisms. Due to the more complicated structure of the payments, the case of the α-VCG mechanism is significantly more challenging.

Lemma 4. *Let $\alpha, \beta \in \mathbb{R}^k_{\geq}$ with $\alpha_k > 0$ and $\beta_k > 0$. Suppose valuations are drawn from a distribution with support $[0, \bar{v}]$, probability density function f, and cumulative distribution function F. Then, an efficient equilibrium of the α-GFP mechanism must use a bidding function b^F with*

$$b^F(v) = \frac{\sum_{s=1}^k \beta_s \int_0^v \frac{dP_s(t)}{dt} t \, dt}{\sum_{s=1}^k \alpha_s P_s(v)}.$$

If b^F is increasing almost everywhere, it constitutes the unique efficient equilibrium. Otherwise no efficient equilibrium exists.

Proof. Since efficient equilibria must be symmetric, we can write an efficient equilibrium of the α-GFP mechanism in terms of a bidding function $b^F : [0, \bar{v}] \to \mathbb{R}_{\geq 0}$. A bidder with value v who is allocated position s then pays $\alpha_s b^F(v)$, and we have that

$$\mathbb{E}\left[p^F(v)\right] = \sum_{s=1}^k \alpha_s P_s(v) b^F(v). \tag{33}$$

The expected payment in an efficient equilibrium is given by Lemma 3, and by equating (33) with (29) and solving for $b^F(v)$ we obtain

$$b^F(v) = \frac{\sum_{s=1}^k \beta_s \int_0^v \frac{dP_s(t)}{dt} t \, dt}{\sum_{s=1}^k \alpha_s P_s(v)}.$$

Bidding below $b^F(0) = 0$ is impossible and bidding above $b^F(\bar{v})$ is dominated, so the claim follows from Lemma 3. □

Lemma 5. *Let $\alpha, \beta \in \mathbb{R}^k_{\geq}$ with $\alpha_k > 0$ and $\beta_k > 0$. Suppose valuations are drawn from a distribution with support $[0, \bar{v}]$, probability density function f, and cumulative distribution function F. Then, an efficient equilibrium of the α-VCG mechanism must use a bidding function b^V with*

$$b^V(v) = \frac{\sum_{s=1}^k \beta_s \frac{dP_s(v)}{dv} v}{\sum_{s=1}^k \alpha_s \frac{dP_s(v)}{dv}}.$$

If b^V is increasing almost everywhere, it constitutes the unique efficient equilibrium. Otherwise no efficient equilibrium exists.

Proof. Efficiency again requires symmetry, so any efficient equilibrium of the α-VCG mechanism can be described by a bidding function $b^V : [0, \bar{v}] \to \mathbb{R}_{\geq 0}$.

Denote by $p^V(v)$ the payment in the α-VCG mechanism of a bidder with value v, and by $p_s^V(v)$ the same payment under the condition that the bidder has the s-highest value overall. These quantities are random variables that depend on the values of $n-1$ other bidders, and we have that

$$\mathbb{E}[p^V(v)] = \sum_{s=1}^{k} P_s(v) \cdot \mathbb{E}[p_s^V(v)], \tag{34}$$

where, as before, $P_s(v)$ is the probability that v is the s-highest of n values drawn independently from F. The conditional payment $p_s^V(v)$ depends on the conditional densities of the valuations of bidders assigned lower positions, and on their resulting bids. For $s \in \{1, \ldots, k\}$ and $\ell \in \{s, \ldots, k\}$, denote by

$$I_{\ell,s}(v,t) = \frac{(n-s) f(t) \binom{n-s-1}{n-\ell-1} F(t)^{n-\ell-1} (F(v) - F(t))^{\ell-s}}{F(v)^{n-s}}$$

the density at t of the $(\ell+1)$-highest of n values drawn independently from F, under the condition that the s-highest value is equal to v. Then

$$\mathbb{E}[p_s^V(v)] = \sum_{\ell=s}^{k} (\alpha_\ell - \alpha_{\ell+1}) \cdot \int_0^v I_{\ell,s}(v,t)\, b^V(t)\, dt,$$

and by substituting into (34), exchanging the order of summation and integration, and grouping by coefficients of α_s, we obtain

$$\begin{aligned} \mathbb{E}[p^V(v)] &= \sum_{s=1}^{k} P_s(v) \sum_{\ell=s}^{k} (\alpha_\ell - \alpha_{\ell+1}) \int_0^v I_{\ell,s}(v,t)\, b^V(t)\, dt \\ &= \int_0^v \sum_{s=1}^{k} \alpha_s \left[\sum_{\ell=1}^{s} P_\ell(v) \cdot I_{s,\ell}(v,t) - \sum_{\ell=1}^{s-1} P_\ell(v) \cdot I_{s-1,\ell}(v,t) \right] b^V(t)\, dt. \end{aligned} \tag{35}$$

Note that the roles of s and ℓ have been reversed, such that $s \geq \ell$ henceforth. We now recall that

$$P_\ell(v) = \binom{n-1}{\ell-1} (1 - F(v))^{\ell-1} F(v)^{n-\ell}$$

and consider each of the two sums inside the parentheses in turn.

Denoting

$$J_{\ell,s} = \binom{n-1}{\ell-1} \binom{n-\ell-1}{n-s-1} (n-\ell),$$

we have that

$$\sum_{\ell=1}^{s} P_\ell(v) \cdot I_{s,\ell}(v,t) = \sum_{\ell=1}^{s} J_{\ell,s} \cdot (1-F(v))^{\ell-1} f(t) F(t)^{n-s-1} (F(v)-F(t))^{s-\ell}$$
$$= \sum_{\substack{1\le \ell \le s\\ 0\le x\le \ell-1\\ 0\le y\le s-\ell}} J_{\ell,s} \binom{\ell-1}{x}\binom{s-\ell}{y} (-1)^{\ell+y-x-1} f(t) F(v)^{s-x-y-1} F(t)^{n+y-s-1},$$

where the second equality holds because by the binomial theorem

$$(1-F(v))^{\ell-1} = \sum_{x=0}^{\ell-1} \binom{\ell-1}{x} (-F(v))^{\ell-x-1} \quad \text{and}$$
$$(F(v)-F(t))^{s-\ell} = \sum_{y=0}^{s-\ell} \binom{s-\ell}{y} F(v)^{s-\ell-y} (-F(t))^{y}.$$

We claim that the terms with $x+y<s-1$ cancel out, i.e., that

$$\sum_{\substack{1\le \ell \le s\\ 0\le x\le \ell-1\\ 0\le y\le s-\ell\\ x+y<s-1}} J_{\ell,s} \binom{\ell-1}{x}\binom{s-\ell}{y} (-1)^{\ell+y-x-1} f(t) F(v)^{s-x-y-1} F(t)^{n+y-s-1}$$
$$= \sum_{\substack{0\le z\le s-2\\ 0\le y\le z\\ z-y+1\le \ell\le s-y}} J_{\ell,s} \binom{\ell-1}{z-y}\binom{s-\ell}{y} (-1)^{\ell+2y-z-1} F(v)^{s-z-1} f(t) F(t)^{n+y-s-1} = 0.$$

Indeed, the first equality follows by setting $z = x+y$ and observing that in both sums ℓ takes exactly the values between $x+1 = z-y+1$ and $s-y$. The second equality holds because for any z and y with $0 \le z \le s-2$ and $0 \le y \le z$,

$$\sum_{\ell=z-y+1}^{s-y} J_{\ell,s} \binom{\ell-1}{z-y}\binom{s-\ell}{y} (-1)^{\ell+2y-z-1}$$
$$= \sum_{\ell=z-y+1}^{s-y} \binom{n-1}{\ell-1}\binom{n-\ell-1}{n-s-1} (n-\ell) \binom{\ell-1}{z-y}\binom{s-\ell}{y} (-1)^{\ell+2y-z-1}$$
$$= \frac{(n-1)!}{(n-s-1)!(z-y)!y!} \sum_{\ell=z-y+1}^{s-y} \frac{(-1)^{\ell+2y-z-1}}{(\ell-z+y-1)!(s-\ell-y)!}$$
$$= \frac{(n-1)!}{(n-s-1)!(z-y)!y!} \sum_{j=0}^{m} \frac{(-1)^{j+y}}{j!(m-j)!}$$
$$= \frac{(n-1)!(-1)^y}{(n-s-1)!(z-y)!y!m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j}$$

$$= \frac{(n-1)!(-1)^y}{(n-s-1)!(z-y)!y!m!}(1+(-1))^m = 0, \tag{36}$$

where the third equality follows by setting $j = \ell - z + y - 1$ and $m = s - z - 1$ and the fifth equality holds by the binomial theorem. Thus, actually,

$$\begin{aligned}
\sum_{\ell=1}^{s} P_\ell(v) \cdot I_{s,\ell}(v,t) &= \sum_{\substack{1\le\ell\le s\\ 0\le x\le \ell-1\\ 0\le y\le s-\ell\\ x+y=s-1}} J_{\ell,s}\binom{\ell-1}{x}\binom{s-\ell}{y}(-1)^{\ell+y-x-1} f(t)F(v)^{s-x-y-1}F(t)^{n+y-s-1}\\
&= \sum_{\ell=1}^{s} J_{\ell,s}\binom{\ell-1}{\ell-1}\binom{s-\ell}{s-\ell}(-1)^{s-\ell} f(t)F(v)^0F(t)^{n-\ell-1}\\
&= \sum_{\ell=1}^{s} J_{\ell,s}\cdot(-1)^{s-\ell}F(t)^{n-\ell-1}f(t)\\
&= \sum_{\ell=1}^{s} \binom{n-1}{s-1}(n-s)\binom{s-1}{\ell-1}(-1)^{s-\ell}F(t)^{n-\ell-1}f(t)\\
&= \sum_{\ell=0}^{s-1} \binom{n-1}{s-1}(n-s)\binom{s-1}{\ell}(-1)^{s-\ell-1}F(t)^{n-\ell-2}f(t)\\
&= \binom{n-1}{s-1}(1-F(t))^{s-1}(n-s)F(t)^{n-s-1}f(t),
\end{aligned} \tag{37}$$

where the third equality holds because

$$\begin{aligned}
J_{\ell,s} &= \binom{n-1}{\ell-1}\binom{n-\ell-1}{n-s-1}(n-\ell) = \frac{(n-1)!}{(n-\ell)!(l-1)!}\,\frac{(n-\ell-1)!}{(s-\ell)!(n-s-1)!}\,(n-\ell)\\
&= \frac{(n-1)!}{(l-1)!(s-\ell)!(n-s-1)!} = \frac{(n-1)!}{(n-s)!(s-1)!}\,\frac{(s-1)!}{(s-\ell)!(\ell-1)!}\,(n-s)\\
&= \binom{n-1}{s-1}\binom{s-1}{\ell-1}(n-s)
\end{aligned}$$

and the fifth equality because by the binomial theorem

$$\sum_{\ell=0}^{s-1}\binom{s-1}{\ell}(-1)^{s-\ell-1}F(t)^{s-\ell-1} = (1-F(t))^{s-1}.$$

Analogously, for the second term in the parentheses of (35),

$$\begin{aligned}
&\sum_{\ell=1}^{s-1} P_\ell(v)\cdot I_{s-1,\ell}(v,t)\\
&\qquad = \sum_{\ell=1}^{s-1} J_{\ell,s-1}\cdot(1-F(v))^{\ell-1} f(t)F(t)^{n-s}(F(v)-F(t))^{s-\ell-1}\\
&\qquad = \sum_{\substack{1\le\ell\le s-1\\ 0\le x\le \ell-1\\ 0\le y\le s-\ell-1}} J_{\ell,s-1}\cdot\binom{\ell-1}{x}\binom{s-\ell-1}{y}(-1)^{\ell+y-x-1} f(t)F(v)^{s-x-y-2}F(t)^{n+y-s},
\end{aligned}$$

where the second equality holds because by the binomial theorem

$$\begin{aligned}
(1-F(v))^{\ell-1} &= \sum_{x=0}^{\ell-1}\binom{\ell-1}{x}(-F(v))^{\ell-x-1}\quad\text{and}\\
(F(v)-F(t))^{s-\ell-1} &= \sum_{y=0}^{s-\ell-1}\binom{s-\ell-1}{y}F(v)^{s-\ell-y-1}(-F(t))^y.
\end{aligned}$$

We claim that the terms with $x+y<s-2$ cancel out, i.e., that

$$\begin{aligned}
&\sum_{\substack{1\le\ell\le s-1\\ 0\le x\le \ell-1\\ 0\le y\le s-\ell-1\\ x+y<s-2}} J_{\ell,s-1}\binom{\ell-1}{x}\binom{s-\ell-1}{y}(-1)^{\ell+y-x-1} f(t)F(v)^{s-x-y-2}F(t)^{n+y-s}\\
&\qquad = \sum_{\substack{0\le z\le s-3\\ 0\le y\le z\\ z-y+1\le\ell\le s-y-1}} J_{\ell,s-1}\binom{\ell-1}{z-y}\binom{s-\ell-1}{y}(-1)^{\ell+2y-z-1} f(t)F(v)^{s-z-2}F(t)^{n+y-s}=0.
\end{aligned}$$

Indeed, the first equality follows by setting $z=x+y$ and observing that in both sums ℓ takes exactly the values between $x+1=z-y+1$ and $s-y-1$. The second equality holds because for any z and y with $0\le z\le s-3$ and $0\le y\le z$,

$$\sum_{\ell=z-y+1}^{s-y-1} J_{\ell,s-1}\binom{\ell-1}{z-y}\binom{s-\ell-1}{y}(-1)^{\ell+2y-z-1} = \sum_{\ell=z-y+1}^{r-y} J_{\ell,r}\binom{\ell-1}{z-y}\binom{r-\ell}{y}(-1)^{\ell+2y-z-1}=0,$$

where the first equality follows by setting $r=s-1$ and the second equality holds by (36). Thus, actually,

$$\begin{aligned}
\sum_{\ell=1}^{s-1} P_\ell(v)\cdot I_{s-1,\ell}(v,t) &= \sum_{\substack{1\le\ell\le s-1\\ 0\le x\le \ell-1\\ 0\le y\le s-\ell-1\\ x+y=s-2}} J_{\ell,s-1}\binom{\ell-1}{x}\binom{s-\ell-1}{y}(-1)^{\ell+y-x-1} f(t)F(v)^{s-x-y-2}F(t)^{n+y-s}\\
&= \sum_{\ell=1}^{s-1} J_{\ell,s-1}\binom{\ell-1}{\ell-1}\binom{s-\ell-1}{s-\ell-1}(-1)^{s-\ell-1} f(t)F(v)^0F(t)^{n-\ell-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{s-1} J_{\ell,s-1} \cdot (-1)^{s-\ell-1} F(t)^{n-\ell-1} f(t) \\
&= \sum_{\ell=1}^{s-1} \binom{n-1}{s-1} (s-1) \binom{s-2}{\ell-1} (-1)^{s-\ell-1} F(t)^{n-\ell-1} f(t) \\
&= \sum_{\ell=0}^{s-2} \binom{n-1}{s-1} (s-1) \binom{s-2}{\ell} (-1)^{s-\ell-2} F(t)^{n-\ell-2} f(t) \\
&= \binom{n-1}{s-1} (1-F(t))^{s-2} (s-1) F(t)^{n-s} f(t), \qquad (38)
\end{aligned}
$$

where the third equality holds because

$$
\begin{aligned}
J_{\ell,s-1} &= \binom{n-1}{\ell-1}\binom{n-\ell-1}{n-s}(n-\ell) = \frac{(n-1)!}{(n-\ell)!(l-1)!}\,\frac{(n-\ell-1)!}{(s-\ell-1)!(n-s)!}\,(n-\ell) \\
&= \frac{(n-1)!}{(l-1)!(s-\ell-1)!(n-s)!} = \frac{(n-1)!}{(n-s)!(s-1)!}\,\frac{(s-2)!}{(s-\ell-1)!(\ell-1)!}\,(s-1) \\
&= \binom{n-1}{s-1}\binom{s-2}{\ell-1}(s-1)
\end{aligned}
$$

and the fifth equality because by the binomial theorem

$$
\sum_{\ell=0}^{s-2} \binom{s-2}{\ell} (-1)^{s-\ell-2} F(t)^{s-\ell-2} = (1-F(t))^{s-2}.
$$

By substituting (37) and (38) into (35), we conclude that

$$
\begin{aligned}
\mathbb{E}\left[p^V(v)\right] &= \int_0^v \sum_{s=1}^{k} \alpha_s \Bigg(\binom{n-1}{s-1} (1-F(t))^{s-1} (n-s) F(t)^{n-s-1} f(t) - \\
&\qquad\qquad \binom{n-1}{s-1} (1-F(t))^{s-2} (s-1) F(t)^{n-s} f(t) \Bigg) b^V(t)\, dt \\
&= \sum_{s=1}^{k} \alpha_s \int_0^v \frac{dP_s(t)}{dt} b^V(t)\, dt. \qquad (39)
\end{aligned}
$$

The expected payment in an efficient equilibrium is again given by Lemma 3. We can thus equate (39) with (29), take derivatives on both sides, and solve for $b^V(v)$ to obtain

$$
b^V(v) = \frac{\sum_{s=1}^{k} \beta_s \frac{dP_s(v)}{dv} v}{\sum_{s=1}^{k} \alpha_s \frac{dP_s(v)}{dv}}.
$$

Bidding below $b^V(0) = 0$ is impossible and bidding above $b^V(\bar{v})$ is dominated, so the claim follows from Lemma 3. $\square$

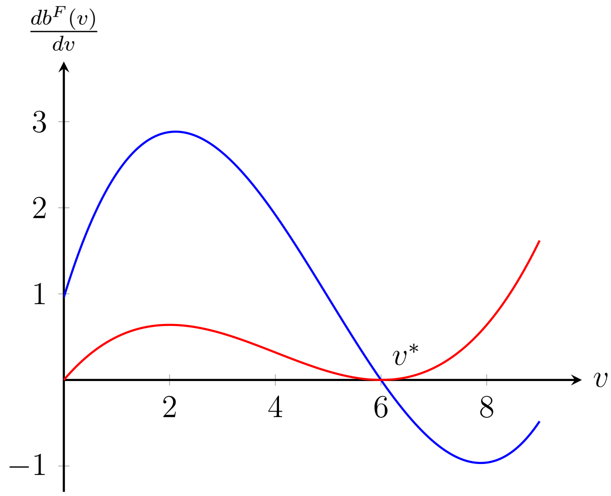

Figure 6: Possible forms of the derivative of the candidate equilibrium bidding function b^F when the α-GFP mechanism does not possess an equilibrium. Both the derivative and the second derivative are non-negative at zero, so if the former is non-positive anywhere on $(0, \bar{v}]$ there must be a value $v^* > 0$ where it either touches or cuts the x-axis from above.

Comparison of the candidate bidding functions Even with the candidate bidding functions b^F and b^V in hand, the cases where the α-GFP and α-VCG mechanisms respectively admit an efficient equilibrium seem difficult to compare. What will ultimately drive the proof of Theorem 2 is a rather curious relationship between the two bidding functions that is straightforward to verify given Lemma 4 and Lemma 5: the numerator of b^V is equal to the derivative of the numerator of b^F, and the denominator of b^V is equal to the derivative of the denominator of b^F.

Corollary 2. *Let $b^F : \mathbb{R} \to \mathbb{R}$ and $b^V : \mathbb{R} \to \mathbb{R}$ be the candidate equilibrium bidding functions for the α-GFP and α-VCG mechanisms as defined in Lemma 4 and Lemma 5. Then*

$$b^F(v) = \frac{A(v)}{B(v)} \quad \textit{and} \quad b^V(v) = \frac{A'(v)}{B'(v)},$$

where $A(v) = \sum_{s=1}^{k} \beta_s \int_0^v \frac{dP_s(t)}{dt} t \, dt$ and $B(v) = \sum_{s=1}^{k} \alpha_s P_s(v)$.

To show that the α-GFP mechanism possesses an efficient equilibrium whenever the α-VCG mechanism does we recall that equilibrium existence is equivalent to a bidding function that is increasing almost everywhere. We will show that whenever the α-GFP mechanism fails to satisfy this property, then the α-VCG mechanism will fail to satisfy this property as well.

We first consider the candidate bidding function for the α-GFP mechanism and show that at $v = 0$, either its derivative is positive or both its derivative and second derivative are non-negative. Failure to possess an equilibrium thus implies existence of a value $v^* > 0$ where the derivative cuts the x-axis from above, or of a set of such values with positive measure where it touches the x-axis. The situation is illustrated in Figure 6.

We show the claimed behavior at $v = 0$ using Lemmas 6 and 7 below. Recall that we can assume without loss of generality that $n \geq k$. If $n > k$, Lemma 6 implies that $db^F(v)/dv \mid_{v=0} > 0$. If $n = k$, Lemma 6 implies that $db^F(v)/dv \mid_{v=0} = 0$, and Lemma 7 shows that $d^2b^F(v)/dv^2 \mid_{v=0} \geq 0$.

Lemma 6. *Let $b^F : \mathbb{R} \to \mathbb{R}$ be the candidate equilibrium bidding function for the α-GFP mechanisms as defined in Lemma 4. Then,*

$$\left.\frac{db^F(v)}{dv}\right|_{v=0} = \frac{n-k}{n-k+1} \cdot \frac{\beta_k}{\alpha_k}.$$

Proof. By Corollary 2, $b^F(v) = A(v)/B(v)$ with $A(v) = \sum_{s=1}^k \beta_s \int_0^v \frac{dP_s(t)}{dt} t \; dt$ and $B(v) = \sum_{s=1}^k \alpha_s P_s(v)$. Writing the derivative as a limit of difference quotients, applying l'Hospital's rule to each of the two resulting terms, and respectively substituting x for 2δ and δ we obtain

$$\begin{aligned}
\left.\frac{db^F(v)}{dv}\right|_{v=0} &= \lim_{\delta\to 0}\left(\frac{A(2\delta)/B(2\delta) - A(\delta)/B(\delta)}{\delta}\right) \\
&= \lim_{\delta\to 0}\frac{A(2\delta)}{\delta\cdot B(2\delta)} - \lim_{\delta\to 0}\frac{A(\delta)}{\delta\cdot B(\delta)} \\
&= \lim_{\delta\to 0}\frac{A'(2\delta)\cdot 2}{\delta\cdot B'(2\delta)\cdot 2 + B(2\delta)} - \lim_{\delta\to 0}\frac{A'(\delta)}{\delta\cdot B'(\delta)+B(\delta)} \\
&= \lim_{x\to 0}\frac{\left(\sum_{s=1}^k \beta_s \frac{dP_s(x)}{dx}\cdot x\right)\cdot 2}{\left(\sum_{s=1}^k \alpha_s \frac{dP_s(x)}{dx}\cdot x\right) + \left(\sum_{s=1}^k \alpha_s P_s(x)\right)} - \\
&\qquad \lim_{x\to 0}\frac{\left(\sum_{s=1}^k \beta_s \frac{dP_s(x)}{dx}\cdot x\right)}{\left(\sum_{s=1}^k \alpha_s \frac{dP_s(x)}{dx}\cdot x\right) + \left(\sum_{s=1}^k \alpha_s P_s(x)\right)}.
\end{aligned}$$

To analyze these limits we extend by $1 = (F(x)^{n-k-1}\cdot x)^{-1}/(F(x)^{n-k-1}\cdot x)^{-1}$, factor $(F(x)^{n-k-1}\cdot x)^{-1}$ into the numerator and denominator, and consider each of the terms in the numerator and denominator in turn.

Using γ as a placeholder for α or β and replacing $P_s(x)$ by its definition,

$$\begin{aligned}
\frac{\sum_{s=1}^k \gamma_s \cdot \frac{dP_s(x)}{dx}\cdot x}{F^{n-k-1}(x)\cdot x} &= \sum_{s=1}^k \gamma_s \Bigg[\binom{n-1}{s-1}(n-s)F^{k-s}(x)(1-F(x))^{s-1}f(x) \\
&\qquad - \binom{n-1}{s-1}(s-1)F^{k-s+1}(x)(1-F(x))^{s-2}f(x)\Bigg] \\
&= \sum_{s=1}^k\sum_{\ell=0}^{s-1}\gamma_s(-1)^\ell\binom{n-1}{s-1}(n-s)\binom{s-1}{\ell}F(x)^{k-s+\ell}f(x) \\
&\qquad - \sum_{s=1}^k\sum_{\ell=0}^{s-2}\gamma_s(-1)^\ell\binom{n-1}{s-1}(s-1)\binom{s-2}{\ell}F(x)^{k-s+\ell+1}f(v).
\end{aligned}$$

Similarly,

$$\frac{\sum_{s=1}^{k}\alpha_s P_s(x)}{F^{n-k-1}(x)\cdot x} = \frac{1}{x}\cdot\sum_{s=1}^{k}\alpha_s\binom{n-1}{s-1}F(x)^{k-s+1}(1-F(x))^{s-1}$$
$$= \frac{F(x)}{x}\cdot\sum_{s=1}^{k}\sum_{\ell=0}^{s-1}\alpha_s(-1)^{\ell}\binom{n-1}{s-1}\binom{s-1}{\ell}F(x)^{k-s+\ell}.$$

Since $\lim_{x\to 0}F(x)^d = 0$ for $d>0$, the only terms that survive in the limit are those where the exponent of $F(x)$ is zero. For $s\in\{1,\dots,k\}$ and $\ell\in\{0,\dots,s-1\}$, $k-s+\ell=0$ only if $s=k$ and $\ell=0$. For $s\in\{1,\dots,k\}$ and $\ell\in\{0,\dots,s-2\}$, $k-s+\ell-1\neq 0$. Using that $\lim_{x\to 0}F(x)/x = f(0)$, we thus obtain

$$\left.\frac{db^F(v)}{dv}\right|_{v=0} = \frac{\beta_k\binom{n-1}{k-1}(n-k)f(0)\cdot 2}{\alpha_k\binom{n-1}{k-1}(n-k)f(0)+\alpha_k\binom{n-1}{k-1}f(0)} - \frac{\beta_k\binom{n-1}{k-1}(n-k)f(0)}{\alpha_k\binom{n-1}{k-1}(n-k)f(0)+\alpha_k\binom{n-1}{k-1}f(0)}$$
$$= \frac{2(n-k)\beta_k}{(n-k+1)\alpha_k} - \frac{(n-k)\beta_k}{(n-k+1)\alpha_k} = \frac{n-k}{n-k+1}\cdot\frac{\beta_k}{\alpha_k}$$

as claimed. □

Lemma 7. *Let $b^F:\mathbb{R}\to\mathbb{R}$ be the candidate equilibrium bidding function for the α-GFP mechanisms as defined in Lemma 4. Then, for $n=k$,*

$$\left.\frac{d^2b^F(v)}{dv^2}\right|_{v=0}\geq 0.$$

Proof. By Corollary 2, $b^F(v) = A(v)/B(v)$ with $A(v)=\sum_{s=1}^{k}\beta_s\int_0^v\frac{dP_s(t)}{dt}t\;dt$ and $B(v)=\sum_{s=1}^{k}\alpha_s P_s(v)$. For $n=k$, by Lemma 6,

$$\left.\frac{db^F(v)}{dv}\right|_{v=0} = \left.\frac{A'(v)B(v)-A(v)B'(v)}{B(v)^2}\right|_{v=0} = 0.$$

Since

$$B(0)=\sum_{s=1}^{k}\alpha_s P_s(0)\geq \alpha_k P_k(0) = \alpha_k(1-F(0))^{n-1} = \alpha_k>0,$$

this implies that

$$\Big(A'(v)B(v)-A(v)B'(v)\Big)\Big|_{v=0} = 0.$$

Thus

$$\left.\frac{d^2b^F(v)}{dv^2}\right|_{v=0} = \left.\frac{(A''(v)B(v)-A(v)B''(v))B(v)^2-(A'(v)B(v)}{B(v)^4}\right|_{v=0} - \left.\frac{A(v)B'(v))2B(v)B'(v)}{B(v)^4}\right|_{v=0}$$
$$= \left.\frac{A''(v)B(v)-A(v)B''(v)}{B(v)^2}\right|_{v=0}.$$

We have already seen that $B(0) > 0$. Moreover, $A(0) = 0$ by the definition of A and $B''(0) < \infty$ by assumption on the value distributions, so it suffices to show that

$$A''(0) = \left(\sum_{s=1}^{k} \beta_s \frac{d^2 P_s(v)}{dv^2} \cdot v + \sum_{s=1}^{k} \beta_s \frac{dP_s(v)}{dv} \right) \Bigg|_{v=0} \geq 0.$$

Also by assumptions on the value distributions, $d^2 P_s(v)/dv^2 < \infty$ for all v, so the first term vanishes. The second term is

$$\begin{aligned} \sum_{s=1}^{k} \beta_s \frac{dP_s(v)}{dv} \Bigg|_{v=0} = \sum_{s=1}^{k} \beta_s \Bigg(\binom{n-1}{s-1} (n-s) F(v)^{n-s-1} (1-F(v))^{s-1} f(v) \\ - \binom{n-1}{s-1} (s-1) F(v)^{n-s} (1-F(v))^{s-2} f(v) \Bigg) \Bigg|_{v=0} \\ = \beta_{k-1}(k-1) f(0) - \beta_k (k-1) f(0) \geq 0, \end{aligned}$$

where we have used the definition of $P_s(v)$ and the fact that the only non-zero terms are those where the exponent of $F(v)$ is zero. Since $\beta_{k-1} \geq b_k$ and $f(0) > 0$, this shows the claim. $\square$

To complete the proof of Theorem 2 we exploit the relationship between the bidding functions of the α-GFP and α-VCG mechanisms established by Corollary 2, and show that the latter inherits its behaviour at values v^* from the former.

Proof of Theorem 2. Assume that the α-GFP mechanism does not possess an efficient equilibrium, and recall that this implies the existence of a set of values with positive measure where the candidate bidding function b^F is not strictly increasing.

By Lemmas 6 and 7, there must thus exist a set of values $v^* > 0$ with positive measure where

$$\frac{db^F(v)}{dv} \Bigg|_{v=v^*} = 0 \qquad \text{and} \qquad \frac{d^2 b^F(v)}{dv^2} \Bigg|_{v=v^*} \leq 0,$$

or one such value where the equality holds and the inequality is strict.

For an arbitrary value v,

$$\frac{db^F(v)}{dv} = \frac{A'(v)B(v) - B'(v)A(v)}{(B(v))^2} = 0$$

requires that

$$A'(v)B(v) - B'(v)A(v) = 0. \tag{40}$$

Assuming (40),

$$\frac{d^2 b^F(v)}{dv^2} = \frac{A''(v)B(v) - B''(v)A(v)}{(B(v))^2} \leq 0$$

requires that

$$A''(v)B(v) - B''(v)A(v) \leq 0, \tag{41}$$

Consider any $v^* > 0$, and observe that $A(v^*) > 0$ and $A'(v^*) > 0$. For $v = v^*$ we can thus rewrite (40) as $B(v^*) = \frac{B'(v^*)A(v^*)}{A'(v)}$, and substitute this into (41) to obtain

$$A''(v^*)\frac{B'(v^*)A(v^*)}{A'(v^*)} - B''(v^*)A(v^*) \leq 0.$$

Dividing by $A(v^*) > 0$ and multiplying by $A'(v^*) > 0$ yields

$$A''(v^*)B'(v^*) - A'(v^*)B''(v^*) \leq 0,$$

and thus

$$\left.\frac{db^V(v)}{dv}\right|_{v=v^*} = \frac{A''(v^*)B'(v^*) - A'(v^*)B''(v^*)}{(B'(v^*))^2} \leq 0.$$

It is, moreover, easily verified that the inequality holds strictly when $d^2b^F(v)/dv^2|_{v=v^*} < 0$. There thus exists a set of values v^* with positive measure where $\frac{db^V(v)}{dv} \leq 0$, and the claim follows. □

A Complete Information, Three Positions, and Four Bidders

Assume without loss of generality that $v_1 \geq v_2 \geq v_3 \geq v_4 > 0$, and in addition that $\beta_1 > \beta_2 > \beta_3 > 0$. Efficiency then requires that

$$b_1 \geq b_2 \geq b_3 \geq b_4. \tag{42}$$

For b to be an equilibrium, none of the bidders may benefit from raising or lowering their respective bid and being assigned a different position, which for the α-VCG mechanism means that

$$\beta_1 v_1 - (\alpha_1 - \alpha_2)b_2 - (\alpha_2 - \alpha_3)b_3 - \alpha_3 b_4 \geq \beta_2 v_1 - (\alpha_2 - \alpha_3)b_3 - \alpha_3 b_4, \tag{43}$$
$$\beta_1 v_1 - (\alpha_1 - \alpha_2)b_2 - (\alpha_2 - \alpha_3)b_3 - \alpha_3 b_4 \geq \beta_3 v_1 - \alpha_3 b_4, \tag{44}$$
$$\beta_1 v_1 - (\alpha_1 - \alpha_2)b_2 - (\alpha_2 - \alpha_3)b_3 - \alpha_3 b_4 \geq 0, \tag{45}$$
$$\beta_2 v_2 - (\alpha_2 - \alpha_3)b_3 - \alpha_3 b_4 \geq \beta_1 v_2 - (\alpha_1 - \alpha_2)b_1 - (\alpha_2 - \alpha_3)b_3 - \alpha_3 b_4, \tag{46}$$
$$\beta_2 v_2 - (\alpha_2 - \alpha_3)b_3 - \alpha_3 b_4 \geq \beta_3 v_2 - \alpha_3 b_4, \tag{47}$$
$$\beta_2 v_2 - (\alpha_2 - \alpha_3)b_3 - \alpha_3 b_4 \geq 0, \tag{48}$$
$$\beta_3 v_3 - \alpha_3 b_4 \geq \beta_1 v_3 - (\alpha_1 - \alpha_2)b_1 - (\alpha_2 - \alpha_3)b_2 - \alpha_3 b_4, \tag{49}$$

$$\beta_3 v_3 - \alpha_3 b_4 \geq \beta_2 v_3 - (\alpha_2 - \alpha_3) b_2 - \alpha_3 b_4, \tag{50}$$
$$\beta_3 v_3 - \alpha_3 b_4 \geq 0, \tag{51}$$
$$0 \geq \beta_1 v_4 - (\alpha_1 - \alpha_2) b_1 - (\alpha_2 - \alpha_3) b_2 - \alpha_3 b_3, \tag{52}$$
$$0 \geq \beta_2 v_4 - (\alpha_2 - \alpha_3) b_2 - \alpha_3 b_3, \tag{53}$$
$$0 \geq \beta_3 v_4 - \alpha_3 b_3. \tag{54}$$

By (46), $(\alpha_1 - \alpha_2) b_1 \geq (\beta_1 - \beta_2) v_2$ and thus $\alpha_1 > \alpha_2$. By (50), $(\alpha_2 - \alpha_3) b_2 \geq (\beta_2 - \beta_3) v_3$ and thus $\alpha_2 > \alpha_3$. By (54), $\alpha_3 b_3 \geq \beta_3 v_4$ and thus $\alpha_3 > 0$. There are no upper bounds on b_1 and no lower bounds on b_4 except $b_4 \geq 0$, and setting b_1 to a large value and $b_4 = 0$ satisfies (46), (49), (51), and (52). It is furthermore easy to see that (45) is implied by (44) and that (48) is implied by (47). Since $(\beta_1 - \beta_3) v_1 - (\alpha_2 - \alpha_3) b_3 \geq (\beta_1 - \beta_3) v_1 - (\beta_2 - \beta_3) v_2 \geq (\beta_1 - \beta_3) v_1 - (\beta_2 - \beta_3) v_1 = (\beta_1 - \beta_2) v_1$, where the first inequality holds because, by (47), $(\alpha_2 - \alpha_3) b_3 \leq (\beta_2 - \beta_3) v_2$, and the second inequality because $v_1 \geq v_2$, (44) is implied by (43). Since $\beta_2 v_4 - \alpha_3 b_3 \leq \beta_2 v_4 - \beta_3 v_4 = (\beta_2 - \beta_3) v_4 \leq (\beta_2 - \beta_3) v_3$, where the first inequality holds because, by (54), $\alpha_3 b_3 \geq \beta_3 v_4$, and the second inequality because $v_3 \geq v_4$, (53) is implied by (50). Since $\alpha_1 - \alpha_2 > 0$, $\alpha_2 - \alpha_3 > 0$, and $\alpha_3 > 0$, we can rewrite the remaining constraints (43), (47), (50), and (54) as upper and lower bounds on b_3 and b_4 and conclude that the α-VCG mechanism possesses an *efficient* equilibrium if and only if there exist bids b_2 and b_3 such that

$$\begin{aligned} \frac{(\beta_1 - \beta_2) v_1}{\alpha_1 - \alpha_2} &\geq b_2 \geq \max\left\{ \frac{(\beta_2 - \beta_3) v_3}{\alpha_2 - \alpha_3}, b_3 \right\}, \\ \frac{(\beta_2 - \beta_3) v_2}{\alpha_2 - \alpha_3} &\geq b_3 \geq \frac{\beta_3 v_4}{\alpha_3}. \end{aligned} \tag{55}$$

Analogously, for the α-GSP mechanism, the equilibrium conditions require that

$$\beta_1 v_1 - \alpha_1 b_2 \geq \beta_2 v_1 - \alpha_2 b_3, \tag{56}$$
$$\beta_1 v_1 - \alpha_1 b_2 \geq \beta_3 v_1 - \alpha_3 b_4, \tag{57}$$
$$\beta_1 v_1 - \alpha_1 b_2 \geq 0, \tag{58}$$
$$\beta_2 v_2 - \alpha_2 b_3 \geq \beta_1 v_2 - \alpha_1 b_1, \tag{59}$$
$$\beta_2 v_2 - \alpha_2 b_3 \geq \beta_3 v_2 - \alpha_3 b_4, \tag{60}$$
$$\beta_2 v_2 - \alpha_2 b_3 \geq 0, \tag{61}$$
$$\beta_3 v_3 - \alpha_3 b_4 \geq \beta_1 v_3 - \alpha_1 b_1, \tag{62}$$
$$\beta_3 v_3 - \alpha_3 b_4 \geq \beta_2 v_3 - \alpha_2 b_2, \tag{63}$$
$$\beta_3 v_3 - \alpha_3 b_4 \geq 0, \tag{64}$$
$$0 \geq \beta_1 v_4 - \alpha_1 b_1, \tag{65}$$
$$0 \geq \beta_2 v_4 - \alpha_2 b_2, \tag{66}$$
$$0 \geq \beta_3 v_4 - \alpha_3 b_3. \tag{67}$$

For (65), (66), (67) it must be the case that $\alpha_1 > 0$, $\alpha_2 > 0$, and $\alpha_3 > 0$, which is weaker than the corresponding condition for the α-VCG mechanism. There are again no upper bounds on b_1, and setting it to a large value satisfies (59), (62), and (65). Since $\beta_3 v_1 - \alpha_3 b_4 \geq \beta_3 v_2 - \alpha_3 b_4 \geq \beta_3 v_3 - \alpha_3 b_4 \geq 0$, where the first two inequalities hold because $v_1 \geq v_2 \geq v_3$ and the third inequality by (64), (58) is implied by (57) and (61) by (60). Since $\beta_2 v_1 - \alpha_2 b_3 \geq \beta_2 v_1 - (\beta_2 - \beta_3) v_2 - \alpha_3 b_4 \geq \beta_2 v_1 - (\beta_2 - \beta_3) v_1 - \alpha_3 b_4 = \beta_3 v_1 - \alpha_3 b_4$, where the first inequality holds because, by (60), $\alpha_2 b_3 \leq (\beta_2 - \beta_3) v_2 + \alpha_3 b_4$, and the second inequality because $v_1 \geq v_2$, (57) is implied by (56). Since $\alpha_1 > 0$, $\alpha_2 > 0$, and $\alpha_3 > 0$, we can rewrite the remaining constraints (56), (60), (63), (64), (66), and (67) as upper and lower bounds on b_2, b_3, and b_4 and conclude that the α-GSP mechanism possesses an *efficient* equilibrium if and only if there exist bids b_2, b_3, and b_4 such that

$$
\begin{aligned}
\frac{(\beta_1 - \beta_2) v_1 + \alpha_2 b_3}{\alpha_1} &\geq b_2 \geq \max\left\{ \frac{(\beta_2 - \beta_3) v_3 + \alpha_3 b_4}{\alpha_2}, \frac{\beta_2 v_4}{\alpha_2}, b_3 \right\}, \\
\frac{(\beta_2 - \beta_3) v_2 + \alpha_3 b_4}{\alpha_2} &\geq b_3 \geq \max\left\{ \frac{\beta_3 v_4}{\alpha_3}, b_4 \right\}, \\
\frac{\beta_3 v_3}{\alpha_3} &\geq b_4.
\end{aligned}
\tag{68}
$$

It is not immediately obvious when these constraints can be satisfied, and why they should in fact be easier to satisfy than the constraints for the α-VCG mechanism. That b_2 and b_3 are each subject to more than one lower bound, and that b_4 affects both the lower bound on b_2 and the upper bound on b_3, seems particularly unpleasant.

In Section 3.2 we established that, even in the general case with an arbitrary number of positions and bidders, the α-GSP mechanism possesses an efficient Nash equilibrium whenever the α-VCG mechanism does. This is achieved by considering a particular, maximal, solution to the constraints for the α-VCG mechanism and mapping it to a solution to the constraints for the α-GSP mechanism. Instead of repeating the argument here, we show strict superiority of the α-GSP mechanism by focusing on the case where $\beta_1 = \alpha_1 = 1$, $\beta_3 = \alpha_3$, and $v_3 = v_4$. By specializing (55) and (68), which requires some work and in the case of the α-VCG mechanism involves showing that one of the resulting lower bounds is always stronger than the other, we see that the α-VCG mechanism possesses an efficient equilibrium if and only if

$$
\frac{\beta_3 (1 - \beta_2) v_1 + (\beta_2 - \beta_3) v_3}{(1 - \beta_2) v_1 + (\beta_2 - \beta_3) v_3} \leq \alpha_2 \leq \frac{(\beta_2 - \beta_3) v_2 + \beta_3 v_3}{v_3},
$$

and the α-GSP mechanism if and only if

$$
\frac{\beta_2 v_3}{(1 - \beta_2) v_1 + \beta_2 v_3} \leq \alpha_2 \leq \frac{(\beta_2 - \beta_3) v_2 + \beta_3 v_3}{v_3}.
$$

The upper bounds are identical in both cases, and it is not difficult to see that the lower bound for the α-GSP mechanism is easier to satisfy than that for the α-VCG mechanism. We compare the bounds in Figure 7, and note that existence of an efficient equilibrium may fail due to over- as well as underestimation of the relative values of the positions.

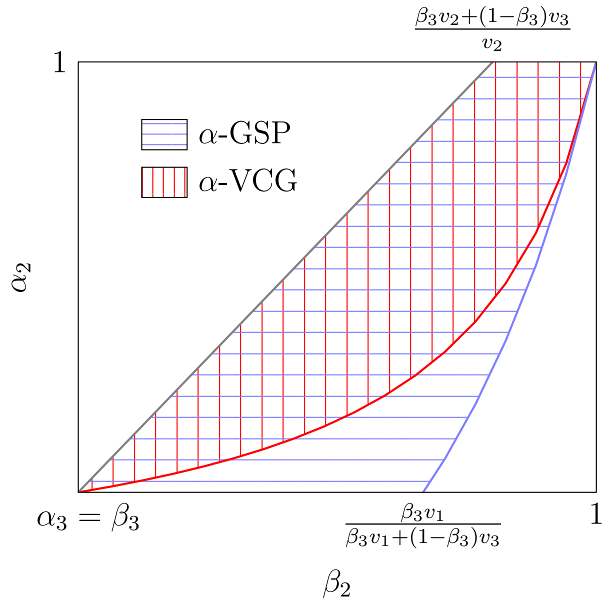

Figure 7: Comparison of the α-GSP and α-VCG mechanisms under complete information, for a setting with three positions and four bidders where $\beta_1 = \alpha_1 = 1$, $\beta_3 = \alpha_3$, and $v_3 = v_4$. The hatched areas indicate the combinations of α_2 and β_2 for which the mechanisms respectively possess an efficient equilibrium. The common upper bound on these areas always starts at the origin and reaches $\alpha_2 = 1$ at $\beta_2 = (\beta_3 v_2 + (1-\beta_3)v_3)/v_2$. The lower bounds for both mechanisms end at the top-right corner. That for the α-GSP mechanism meets the horizontal axis at $\beta_2 = \beta_3 v_1/(\beta_3 v_1 + (1-\beta_3)v_3)$, whereas that for the α-VCG mechanism starts at the origin and curves more strongly toward the bottom-right corner as v_3 decreases.

References

[1] Z. Abrams, A. Ghosh, and E. Vee. Cost of conciseness in sponsored search auctions. In *Proceedings of the 3rd International Workshop on Internet and Network Economics*, pages 326–334, 2007.

[2] Alphabet Inc. Annual report 2017. `https://abc.xyz/investor/pdf/20171231_alphabet_10K.pdf`, 2018. Accessed June 1, 2018.

[3] L. M. Ausubel and P. Milgrom. The lovely but lonely Vickrey auction. In P. Cramton, Y. Shoham, and P. Steinberg, editors, *Combinatorial Auctions*, chapter 1, pages 17–40. MIT Press, 2006.

[4] M. Babaioff and T. Roughgarden. Equilibrium efficiency and price complexity in sponsored search auctions. In *Proceedings of the 6th Workshop on Ad Auctions*, 2010.

[5] Xiaohui Bei, Nick Gravin, Pinyan Lu, and Zhihao Gavin Tang. Correlation-robust analysis of single item auction. In *Proceedings of the 30th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 193–208, 2019.

[6] D. Bergemann and K. Schlag. Robust monopoly pricing. *Journal of Economic Theory*, 146(6):2527–2543, 2011.

[7] L. Blumrosen, J. Hartline, and S. Nong. Position auctions and non-uniform conversion rates. In *Proceedings of the 4th Workshop on Ad Auctions*, 2008.

[8] G. E. P. Box. Science and statistics. *Journal of the American Statistical Association*, 71(356):791–799, 1976.

[9] I. Caragiannis, C. Kaklamanis, P. Kanellopoulos, M. Kyropoulou, B. Lucier, R. Paes Leme, and É. Tardos. Bounding the inefficiency of outcomes in generalized second price auctions. *Journal of Economic Theory*, 156:343–388, 2015.

[10] V. Carrasco, V. Farinha Luz, N. Kos, M. Messner, P. Monteiro, and H. Moreira. Optimal selling mechanisms under moment conditions. *Journal of Economic Theory*, 177:245–279, 2018.

[11] Gabriel Carroll. Robustness and separation in multidimensional screening. *Econometrica*, 85(2):453–488, 2017.

[12] S. Chawla and J. D. Hartline. Auctions with unique equilibria. In *Proceedings of the 14th ACM Conference on Electronic Commerce*, pages 181–196, 2013.

[13] S. Chawla, J. Hartline, and D. Nekipelov. Mechanism design for data science. In *Proceedings of the 15th ACM Conference on Economics and Computation*, pages 711–712, 2014.

[14] P. Dütting, F. Fischer, and D. C. Parkes. Simplicity-expressiveness tradeoffs in mechanism design. In *Proceedings of the 12th ACM Conference on Electronic Commerce*, pages 341–350, 2011.

[15] P. Dütting, F. Fischer, and D. C. Parkes. Expressiveness and robustness of first-price position auctions. *Mathematics of Operations Research*, 44(1):196–211, 2019.

[16] Paul Dütting and Thomas Kesselheim. Posted pricing and prophet inequalities with inaccurate priors. In *Proceedings of the 20th ACM Conference on Electronic Commerce*, pages 111–129, 2019.

[17] Paul Dütting, Tim Roughgarden, and Inbal Talgam-Cohen. Simple versus optimal contracts. In *Proceedings of the 20th ACM Conference on Electronic Commerce*, pages 369–387, 2019.

[18] B. Edelman and M. Ostrovsky. Strategic bidder behavior in sponsored search auctions. *Decision Support Systems*, 43(1):192–198, 2007.

[19] B. Edelman, M. Ostrovsky, and M. Schwartz. Internet advertising and the generalized second price auction: Selling billions of dollars worth of keywords. *American Economic Review*, 97(1):242–259, 2007.

[20] R. Gomes and K. S. Sweeney. Bayes-Nash equilibria of the generalized second-price auction. *Games and Economic Behavior*, 86:421–437, 2014.

[21] Google Inc. The Google AdWords Help Center, 2018. URL `https://support.google.com/adwords`. Accessed June 18, 2018.

[22] T. Graepel, J. Q. Candela, T. Borchert, and R. Herbrich. Web-scale Bayesian click-through rate prediction for sponsored search advertising in Microsoft's Bing search engine. In *Proceedings of the 27th International Conference on Machine Learning*, pages 13–20, 2010.

[23] Nick Gravin and Pinyan Lu. Separation in correlation-robust monopolist problem with budget. In *Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2069–2080, 2018.

[24] D. Hoy, K. Jain, and C. A. Wilkens. A dynamic axiomatic approach to first-price auctions. In *Proceedings of the 14th ACM Conference on Electronic Commerce*, page 583, 2013.

[25] T. Kaplan and S. Zamir. Asymmetric first-price auctions with uniform distributions: Analytic solutions to the general case. *Economic Theory*, 50(2):269–302, 2012.

[26] H. B. Leonard. Elicitation of honest preferences for the assignment of individuals to positions. *Journal of Political Economy*, 91(3):461–479, 1983.

[27] K. Madarász and A. Prat. Sellers with misspecified models. *Review of Economic Studies*, 84(2):790–815, 2017.

[28] Eric Maskin and John Riley. Asymetric auctions. *Review of Economic Studies*, 67(3): 413–438, 2000.

[29] H. B. McMahan, G. Holt, D. Sculley, M. Young, D. Ebner, J. Grady, L. Nie, T. Phillips, E. Davydov, D. Golovin, S. Chikkerur, D. Liu, M. Wattenberg, A. M. Hrafnkelsson, T. Boulos, and J. Kubica. Ad click prediction: A view from the trenches. In *Proceedings of the 19th International Conference on Knowledge Discovery and Data Mining*, pages 1222–1230, 2013.

[30] Microsoft Corp. Bing Ads Help, 2018. URL `https://help.bingads.microsoft.com`. Accessed June 18, 2018.

[31] P. Milgrom. Simplified mechanisms with an application to sponsored-search auctions. *Games and Economic Behavior*, 70(1):62–70, 2010.

[32] R. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58—73, 1981.

[33] M. H. Rothkopf. Thirteen reasons why the Vickrey-Clarke-Groves process is not practical. *Operations Research*, 55(2):191–197, 2007.

[34] H. Varian. Position auctions. *International Journal of Industrial Organization*, 25(6): 1163–1178, 2007.

[35] H. R. Varian and C. Harris. The VCG auction in theory and practice. *American Economic Review*, 104(5):442–445, 2014.

[36] W. Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance*, 16(1):8–37, 1961.