

# Faster Symmetric Rendezvous on Four or More Locations

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## Abstract

In the symmetric rendezvous problem two players follow the same (randomized) strategy to visit one of  $n$  locations in each time step  $t = 0, 1, 2, \dots$ . Their goal is to minimize the expected time until they visit the same location and thus meet. Anderson and Weber [J. Appl. Prob., 1990] proposed a strategy that operates in rounds of  $n - 1$  steps: a player either remains in one location for  $n - 1$  steps or visits the other  $n - 1$  locations in random order; the choice between these two options is made with a probability that depends only on  $n$ . The strategy is known to be optimal for  $n = 2$  and  $n = 3$ , and there is convincing evidence that it is *not* optimal for  $n = 4$ . We show that it is *not* optimal for any  $n \geq 4$ , by constructing a strategy with a smaller expected meeting time.

## 1 Introduction

In 1976, Steve Alpern gave a talk at the Vienna Institute for Advanced Studies where he proposed the following innocent-looking problem [cf. Alpern, 2002]:

In each of two rooms, there are  $n$  telephones randomly strewn about. They are connected in a pairwise fashion by  $n$  wires. At discrete times  $t = 0, 1, 2, \dots$  players in each room pick up a phone and say ‘hello.’ They wish to minimize the time  $t$  when they first pick up paired phones and can communicate.

This problem, which can alternatively be thought of in terms of two individuals trying to meet in one of  $n$  physical locations such as rooms in a house or shops in a mall, has become known as the rendezvous problem on discrete locations. Since there is nothing in the description of the problem that distinguishes the telephones or locations, it is common to assume that they are given a labeling for each player that is chosen uniformly at random and independently from the labeling used by the other player. An optimal strategy then is one that minimizes the expected meeting time, where the expectation is taken over the random labelings and any randomness in players’ strategies. Whether a distinction can be made between the players leads to two natural versions of the problem. The asymmetric version, where players are allowed to use distinct strategies, is straightforward. In the optimal pair of strategies, called wait-for-mommy, one player stays in the same location throughout while the other tours all locations in random order. This leads to an expected meeting time of  $(n - 1)/2$ . In the symmetric version, both players must use the same strategy. A strategy that like wait-for-mommy assigns distinct roles to the players hence cannot

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be used, and any deterministic strategy will with significant probability chase itself forever and therefore has an infinite expected meeting time.

It is natural to exploit what we know about the asymmetric problem, while using randomness to break symmetry. As our objective will be to prove upper bounds on the optimal meeting time, we may assume that locations at time 0 are chosen uniformly at random; we will call the location a player visits at time 0 her *home location*. We can then focus on strategies that start at time 1 and have access to a home location guaranteed to be distinct from the home location of the other player. [Anderson and Weber \[1990\]](#) proposed a strategy, now called the Anderson–Weber strategy or AW for short, in which players repeatedly choose between waiting and touring: with some probability  $\theta$  a player stays in her home location for  $n - 1$  steps, while with probability  $(1 - \theta)$  she tours the remaining  $n - 1$  locations in random order. For  $n = 2$  and  $\theta = 1/2$ , this strategy just chooses a random location in each step, and it is not difficult to show that this is optimal [[Anderson and Weber, 1990](#)]. [Anderson and Weber](#) also claimed that their strategy is optimal for  $n = 3$ , but their proof turned out to be flawed. Optimality for  $n = 3$  was finally shown by [Weber \[2012\]](#). The difficult proof proceeds by showing that the matrix of meeting times for an easier problem, truncated after  $k$  steps for any  $k$ , can be rounded down to a matrix which is positive semidefinite and for which AW is optimal.

The simple description of the symmetric rendezvous problem belies significant difficulty, which not only concerns the search for good strategies but also some very basic properties. [Anderson and Weber](#) conjectured for example that the optimal meeting time must increase with the number of locations, as it does in the asymmetric version, but this conjecture is still open. It also seems clear that the symmetric version is more difficult than the asymmetric one, but this was only shown very recently and only for a difference in meeting times of  $2^{-36}$  [[Bonamy, Pilipczuk, Sereni, and Weber, 2023](#)]. Better lower bounds on the meeting time only exist for  $n \leq 5$ , where computational techniques can be used [[Fan, 2009](#)], and for  $n \rightarrow \infty$  [[Dani, Hayes, Moore, and Russell, 2016](#)]. Optimal strategies for  $n \geq 4$ , and the conjecture of [Anderson and Weber](#) that AW is optimal for  $n \rightarrow \infty$ , currently seem out of reach.

## 1.1 Our Contribution

We give a strategy, which we call First-Same-or-Different and abbreviate FSD, that has a smaller expected meeting time than AW for any  $n \geq 4$ . [Anderson and Weber \[1990\]](#) had conjectured, albeit without justification, that an improvement would be possible for  $n = 4$ . [Fan \[2009\]](#), on the other hand, conjectured optimality of AW for all  $n$  and based this on computational evidence. An unpublished manuscript of [Weber \[2009\]](#) finally gave concrete evidence that an improvement is possible for  $n = 4$ . [Weber](#) modifies AW over blocks of 12 consecutive steps, corresponding to four blocks of  $n - 1 = 3$  steps each in which a player tours; in the first two blocks locations are visited in random order as in AW, but orders in the latter two blocks are correlated. The particular strategy was derived from a negative eigenvalue of a matrix of meeting times, using intuition gained for  $n = 3$  on the relationship between positive semidefiniteness and optimality of AW.

[Weber](#) suggests that the improvement over AW can be seen most easily by calculating the meeting time for  $1585^2$  pairs of sequences of locations, and multiplying it with the probability that a particular pair occurs. These calculations are not made explicit but are feasible with the help of a computer. When it comes to sub-optimality of AW for  $n = 4$  this argument is convincing enough, but if our goal is to improve on AW for general values of  $n$ , and perhaps in the limit, it is a bit of a dead end. Computing the meeting time explicitly for an equally complicated strategy

when  $n > 4$  is out of the question. More importantly, the computations underlying Weber’s choice of improving strategy also become infeasible for  $n > 4$ , leaving us without a candidate strategy.

In AW, and AW-like strategies like that of Weber, players choose a permutation in each round and meet if the two permutations have at least one fixed point, i.e., if they are *not* derangements.<sup>1</sup> It is therefore a bit surprising that the literature on the rendezvous problem has featured only fairly simple results on derangements, like the classic one that two random permutations of length  $n$  are derangements with a probability that approaches  $1/e$  as  $n \rightarrow \infty$ . We will use intuition, and somewhat deeper results, on the combinatorics of derangements to construct and analyze a strategy that improves over AW. Like Weber, we take AW as a starting point, operate in rounds of length  $n - 1$  equal either to the player’s home location or to a permutation of all locations except the home location, and introduce correlation among consecutive permutations. Specifically, we consider permutations in groups of three and require that their respective first elements are either all the same or all different; subject to this constraint, permutations are chosen uniformly at random. This is well-defined for all values of  $n$ , and we will see that it leads to an improvement over AW for all  $n \geq 4$ . Denoting Anderson–Weber’s and our strategy for a specific value of the parameter  $\theta$  by  $AW(\theta)$  and  $FSD(\theta)$ , respectively, our main result is the following.

**Theorem 2 (informal).** *Let  $n \in \mathbb{N}$  and  $\theta \in [0, 1)$ . The expected meeting time of  $FSD(\theta)$  is smaller than the expected meeting time of  $AW(\theta)$  by  $\frac{8(1-\theta)^6}{81(8-(3\theta^2-2\theta+1)^3)}$  if  $n = 4$ , and by  $\frac{483(1-\theta)^6}{(n-1)^8}$  if  $n \geq 5$ .*

For the values of  $\theta$  that are optimal for  $AW(\theta)$ , we obtain an improvement of 0.00125 for  $n = 4$  and of 0.00086 for  $n = 5$ . This yields respective improvements of 0.000938 and 0.00069 compared to the expected meeting times of 2.5685 and 3.3793 of AW for the original problem where players meet at time 0 with probability  $1/n$ .<sup>2</sup>

To see why the type of correlation we use might be helpful, we can view round-based strategies like AW as being played on a graph, where vertices represent permutations and two vertices are connected by an edge if they are *not* derangements. This is the complement of the so-called *derangement graph* studied in combinatorics [e.g. Meagher, Razafimahatratra, and Spiga, 2021, Renteln, 2007, Rasmussen and Savage, 1994]. When a player tours the other locations, AW ignores the structure of the graph and repeatedly chooses a vertex uniformly at random. Our strategy chooses three vertices that either belong to a large clique or to three distinct large cliques; the choice between these two options is made randomly with a probability that depends only on  $n$ . This can be seen as replicating on cliques of permutations what AW does on locations, and one may hope that it improves over AW in the same way in which AW improves over a strategy that in each step chooses a random location. Whether it actually provides an improvement depends on the clique structure, and showing that it does so for our particular choice of strategy requires a fairly intricate analysis. The analysis is complicated further by the fact that the permutations used by the two players are not of the same set of locations but of sets that differ by one element, owing to the fact that players do not visit their home location when moving. Moreover, for the expected meeting time, it not only matters whether two permutations have a fixed point but also where in the permutations the first fixed point occurs.

Fan [2009] and Alpern [2013] contemplate that a good symmetric strategy for rendezvous could be published in a survival guide that one could consult if one needed to meet in an unknown

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<sup>1</sup>Here and in the following we call two permutations derangements if they do not share a position with the same entry. A single permutation is called a derangement if it is a derangement of the identity.

<sup>2</sup>These values differ by 1 from those given by Anderson and Weber [1990], who count time steps starting from 1.

environment. While its analysis is fairly involved, our strategy is simple enough to be published in such a guide. The only change compared to AW affects rounds divisible by three, when we have moved for the previous two rounds and decide to move again; if in the previous two rounds we started in the same location, we again start in that location; if we started in distinct locations, we start in a location that is distinct from both; if the tour we have sampled for the current round does not satisfy this criterion, we sample it again until it does.

## 1.2 Structure of the Paper

Our analysis of FSD relies heavily on the following intermediate result, stated as Theorem 1 in Section 4.1: for  $n \geq 5$ , FSD has a higher probability of meeting than AW after  $r$  rounds for any  $r \geq 3$ ; for  $n = 4$  the probabilities are equal. It follows from the proof of the optimality of AW for  $n = 3$  of Weber [2012] that AW maximizes the meeting probability after  $k$  steps for all  $k$ . To gain intuition and illustrate the key ideas of the proof, we prove a simplified version of this result in Section 3, concerning variants of AW and FSD that visit all  $n$  locations in each round. We then prove Theorem 1 by expressing the probability of meeting within the first three rounds, under FSD and under the condition that both players move in all of these rounds, in terms of the proportion of *shifted derangements*. Shifted derangements generalize the notion of derangements to pairs of permutations of distinct sets; a formal definition, and a bound on their proportion among the set of all permutations that is used in the proof of the theorem, are given in Section 2.2.

Our main result, concerning the expected meeting time under FSD, is shown in Section 4.2. The key ingredient of the proof, along with Theorem 1, is Lemma 6, which gives an improvement in expected meeting time under the condition that the players move in the first three rounds and meet in the third round. To prove the lemma we further condition on the respective first locations the players visit in the third round and compute, for both AW and FSD, the conditional expectations and associated conditional probabilities. The conditional expectations are the same for both strategies and are given in Lemma 8; the key ingredient for their computation is an explicit expression for the expected index of the first fixed point of a permutation taken uniformly at random from  $\mathcal{P}(\{1+b, \dots, n+b\})$ , conditional on it having at least one fixed point (Lemma 7). Regarding the conditional probabilities we show that, under the condition that players move for all of the first three rounds and meet for the first time in the third round, FSD has a larger bias towards the following: (i) the players meeting in the first step of the third round, which leads to best-possible meeting time under the given conditions; and (ii) both players visiting the home location of the other player in the first step of the third round, which leads to an earlier meeting time later in the round compared to the case where players visit distinct locations in the first step that are not home locations. Computing the conditional probabilities for FSD is non-trivial due to the correlations between different rounds; we first compute the joint probabilities with which the players fail to meet in the first two rounds and visit a certain pair of locations in the first step of the third round (Lemma 9) and then apply Bayes' rule.

The fact that Theorem 2 follows from Theorem 1 and Lemma 6 is intuitive but non-trivial. To prove it, we use that both AW and FSD restart every  $3(n-1)$  steps to write the unconditional expectations in terms of expectations conditioned on either (i) meeting within the first two rounds or (ii) meeting in the third round. In the first case conditional expectations are the same under both strategies, and we bound their difference in the second case. For  $n = 4$  we exploit the fact that the meeting probabilities within each round are the same under both strategies. For  $n \geq 5$  we show that, conditioned on meeting in the third round, (i) the expected meeting time is lower

when both players move for all three rounds than in the other cases and (ii) this event has a higher probability under FSD than under AW by Theorem 1.

### 1.3 Open Questions

Our main objective at this point has been to obtain an improved rather than an optimal strategy, and indeed we do not believe that our strategy is optimal. We have therefore not made an attempt to optimize the parameters of our strategy, but there are two obvious ways in which it can be improved. The first is to optimize the probability  $\theta$  of staying in the home location; this probability will be different from the corresponding optimal probability for AW. The second concerns the set of rounds for which our strategy deviates from AW. We have chosen to only modify AW in every third round, under the condition that the player moves in that round and in the two rounds preceding it. However, since players meet with probability one in a round where exactly one player moves, the moving rounds of both players always coincide; we could thus have applied the same definition to every third *moving* round. Both changes clearly lead to an improvement, but significantly complicate the analysis. The question of optimal strategies for  $n \geq 4$  is wide open.

It is clear from the statement of our main result that the advantage of FSD over AW vanishes as  $n \rightarrow \infty$ . The conjecture that AW is optimal in the limit thus remains unresolved, but we believe that an appropriate generalization of our strategy could provide a path toward disproving the conjecture. This will not be straightforward. Specifically, the aforementioned optimization of  $\theta$  and the set of modified rounds do not lead to an improvement in the limit. The same is true for the obvious generalization from blocks of 3 to blocks of  $(n-1)$  moving rounds. The intuition underlying our definition of FSD suggests other generalizations. As we will see later, FSD can be thought of as operating on cliques of the complement of the derangement graph, and makes particular choices corresponding to the contraction of certain cliques and the way in which cliques are visited. We have made these choices in a good way but not obviously in an optimal way, and we were guided to a certain extent by our ability to analyze the resulting strategy. It is possible that the choices can be made in a way that leads to an improvement in the limit, but this is likely to require a better understanding of the structure of the derangement graph and substantial technical innovation.

### 1.4 Further Related Work

Rendezvous problems have been discussed informally many times, for example by Schelling [1960] for two parachutists who have landed in a field and by Mosteller [1965] for two strangers wanting to meet in New York City. The particular problem we study here was first stated formally by Alpern in 1976 [cf. Alpern, 2002], as a more tractable discrete version of the astronaut problem, where players try to meet on the surface of a sphere. No significant progress seems to have been made on the astronaut problem, and the intuition about the relative difficulty of these two problems was most certainly correct. But as we have already discussed, discrete rendezvous problems are anything but tractable. Another notorious example places the two players on the real line, at distance two and without a common orientation [Alpern, 1995]. While this problem was formulated as a continuous one, it turns out to be discrete since strategies where players move at maximum speed and change direction only at unit time steps dominate all other strategies [Han, Du, Vera, and Zuluaga, 2008]. The asymmetric version is again not very difficult, and has an optimal meeting time of 3.25 [Alpern and Gal, 1995]. It is worth noting that in the optimal pair of strategies one player remains more stationary than the other, but not completely stationary. The optimal meeting time

for the symmetric version lies in the interval  $(4.1520, 4.2574)$ ; both the upper and the lower bound were obtained using semidefinite programming. The problem has also been studied for the case where the initial distance is unknown and the goal is to optimize the competitive ratio between the meeting time and half the initial distance [Baston and Gal, 1998]. The competitive is at most 11.028 for asymmetric strategies [Alpern and Beck, 2000], which has been conjectured to be optimal, and at most 13.926 for symmetric strategies [Klimm, Sagnol, Skutella, and Tran, 2022]. The problem where only one player moves, known as the cow path problem, is somewhat more tractable. Here optimal deterministic and randomized strategies are known, with competitive ratios 9 [Baeza-Yates, Culberson, and Rawlins, 1993] and 4.591 [Kao, Reif, and Tate, 1996], respectively.

The standard objective in rendezvous search is minimizing the expected meeting time, but Dani et al. [2016] have studied our problem with the goal of maximizing the probability of meeting after at most  $k$  steps for some  $k$ , in the limit as  $n \rightarrow \infty$ . It turns out that AW is optimal in this respect when  $k \leq n$  but far from optimal when  $k \geq 4n$ ; indeed, while AW fails to meet with constant probability after any fixed number of steps, there exists a strategy that meets almost surely after  $4n$  steps. The analysis of meeting probabilities has also led to a lower bound on the expected meeting time of  $0.6389n$ , again as  $n$  grows; the best known strategy for this case is AW with a meeting time of around  $0.8289n$ .

Alpern, Baston, and Essegaier [1999] have considered a generalization where the game is played on a graph and players can only move to adjacent vertices.<sup>3</sup> The absence of some edges makes the problem harder by restricting travel but may provide opportunities for coordination. It clearly offers an advantage if the graph is *not* vertex-transitive, by allowing players to restrict attention to a subset of the vertices, but even for vertex-transitive graphs it can make the problem significantly easier at least with respect to meeting probabilities [Dani et al., 2016].

A sizeable literature exists on continuous problems, and problems on the line in particular. Alpern [2011] provides a detailed overview of the area with many open questions.

## 2 Preliminaries

Let  $\mathbb{N}$  denote the positive integers and  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . For  $n \in \mathbb{N}$ , let  $[n] := \{1, 2, \dots, n\}$  be the set of integers from 1 to  $n$ . For a set  $S$ , let  $\mathcal{P}(S)$  be the set of permutations of  $S$ , understood as bijections  $\pi: [|S|] \rightarrow S$  mapping indices in  $\{1, \dots, |S|\}$  to distinct elements of  $S$ .

A *strategy* for the symmetric rendezvous problem on  $n$  locations is a distribution over infinite sequences of elements of  $[n]$ . We will often also use the term *strategy* to refer to a family of such distributions, one for each value of  $n$ , and may omit the dependence on  $n$  when its value is clear from context. We will denote sequences by functions  $s: \mathbb{N}_0 \rightarrow [n]$  and strategies by distributions  $\sigma$  over such functions. We will assume for simplicity that both players apply the same strategy according to their own labeling of the  $n$  locations, and that the bijection between their two labelings is drawn uniformly at random. The *meeting time* of a strategy  $\sigma$  is then

$$T_\sigma := \min\{t \in \mathbb{N}_0: \pi(s^1(t)) = s^2(t)\},$$

where  $s^1$  and  $s^2$  are sequences drawn independently from  $\sigma$  and  $\pi$  is a permutation drawn uniformly at random from  $\mathcal{P}([n])$ .

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<sup>3</sup>Note that this is a different graph from the one we have considered earlier, where players meet when they visit adjacent vertices.

## 2.1 Home Location and Round-based Strategies

We only consider strategies that choose the first location uniformly at random, independently of what follows. Let  $\sigma$  be such a strategy, and let  $t_\sigma$  be the random variable equal to the number of steps until rendezvous, assuming this number is at least one, i.e.,  $t_\sigma := T_\sigma \mathbb{1}_{T_\sigma > 1}$ . Then

$$\mathbb{P}[T_\sigma \leq t^*] = \frac{1}{n} + \frac{n-1}{n} \mathbb{P}[t_\sigma \leq t^*] \quad \text{for all } t^* \in \mathbb{N}, \text{ and} \quad \mathbb{E}[T_\sigma] = \frac{n-1}{n} \mathbb{E}[t_\sigma].$$

We will henceforth use  $t_\sigma$ , with the knowledge that the players start at distinct locations at time 0. For simplicity, we will further assume a *universal labeling* of the locations under which player  $i \in \{1, 2\}$  starts at location  $i$  at time 0 and then visits a (random) sequence  $s^i: \mathbb{N} \rightarrow [n]$  of locations. We will refer to the location  $i$  at which player  $i$  starts as the player's *home location*. That the strategies we use are symmetric will be readily appreciated from their definition.

We will focus on *round-based strategies* with rounds of length  $\ell$  for some  $\ell \in \mathbb{N}$ , or  $\ell$ -*strategies* for short. The time steps between  $(r-1)\ell + 1$  and  $r\ell$  for  $r \in \mathbb{N}$  will be called the  $r$ th *round*. Players decide at the beginning of a round whether they stay at their home location for the entire round or move, possibly in a random way, across locations; this process is repeated until rendezvous. More formally, player  $i \in \{1, 2\}$  chooses a sequence  $\chi_1^i, \chi_2^i, \dots$ , where  $\chi_r^i \in \{S, M\}$  for each  $r \in \mathbb{N}$ , and a sequence  $\pi_1^i, \pi_2^i, \dots$ , where  $\pi_r^i: [\ell] \rightarrow [n]$  is a sequence of length  $\ell$  for each  $r \in \mathbb{N}$ . If  $\chi_r^i = S$ , then  $s^i((r-1)\ell + j) = i$  for all  $j \in [\ell]$ , i.e., player  $i$  *stays* at her home location during the  $r$ th round. If  $\chi_r^i = M$ , then  $s^i((r-1)\ell + j) = \pi_r^i(j)$  for every  $j \in [\ell]$ , i.e., player  $i$  *moves* across locations in an order given by  $\pi_r^i$ . A particular round-based strategy is completely defined by the round length  $\ell$  and the distributions from which the sequences  $\chi_1^i, \chi_2^i, \dots$  and  $\pi_1^i, \pi_2^i, \dots$  are drawn.

Anderson and Weber [1990] proposed an  $(n-1)$ -strategy with parameter  $\theta \in [0, 1]$ , which we will denote  $\text{AW}(\theta)$ , where

- (i)  $\chi_r^i = S$  with probability  $\theta$  and  $\chi_r^i = M$  with probability  $1 - \theta$  for each  $i \in \{1, 2\}$  and  $r \in \mathbb{N}$ , independently of all other rounds; and
- (ii)  $\pi_r^i$  is a permutation taken uniformly at random from  $\mathcal{P}([n] \setminus \{i\})$  for each  $i \in \{1, 2\}$  and  $r \in \mathbb{N}$ , independently of all other rounds.

With the goal of reducing the expected meeting time we define a different  $(n-1)$ -strategy, which we will call *First-Same-or-Different* with parameter  $\theta \in [0, 1]$  and denote  $\text{FSD}(\theta)$ , where

- (i)  $\chi_r^i = S$  with probability  $\theta$  and  $\chi_r^i = M$  with probability  $1 - \theta$  for each player  $i \in \{1, 2\}$  and round  $r \in \mathbb{N}$ , independently of all other rounds; and
- (ii) for each  $i \in \{1, 2\}$ ,  $\pi_r^i$  is a permutation taken uniformly at random from  $\mathcal{P}([n] \setminus \{i\})$  for each round  $r \in \mathbb{N}$  with  $r/3 \notin \mathbb{N}$ , independently of all other rounds, and  $\pi_r^i$  is a permutation taken uniformly at random from the set

$$\begin{array}{ll} \{\pi \in \mathcal{P}([n] \setminus \{i\}) : \pi(1) = \pi_{r-2}^i(1)\} & \text{if } \chi_{r-2}^i = \chi_{r-1}^i = \chi_r^i = M, \pi_{r-2}^i(1) = \pi_{r-1}^i(1), \\ \{\pi \in \mathcal{P}([n] \setminus \{i\}) : \pi(1) \notin \{\pi_{r-2}^i(1), \pi_{r-1}^i(1)\}\} & \text{if } \chi_{r-2}^i = \chi_{r-1}^i = \chi_r^i = M, \pi_{r-2}^i(1) \neq \pi_{r-1}^i(1), \\ \mathcal{P}([n] \setminus \{i\}) & \text{otherwise,} \end{array}$$

for each  $r \in \mathbb{N}$  with  $r/3 \in \mathbb{N}$ .

For both strategies, in each round, the players stay at their home location with probability  $\theta$  and tour all non-home locations with the remaining probability  $1 - \theta$ . In  $\text{AW}(\theta)$ , tours are taken independently and uniformly at random for each round where a player moves.  $\text{FSD}(\theta)$  differs from  $\text{AW}(\theta)$  only in rounds that are multiples of three, and only if a player moves in this and the previous two rounds. In this case, if the previous two permutations start with the same location,  $\text{FSD}(\theta)$

chooses the third permutation uniformly at random from those also starting with this common location; if the previous two permutations start with distinct locations,  $\text{FSD}(\theta)$  chooses the third permutation uniformly at random from those starting with a location that differs from both. We note that this is well defined when the number of locations is at least four.

## 2.2 Shifted Derangements

For a permutation  $\pi$ , call  $i \in [n]$  such that  $\pi(i) = i$  a *fixed point* of  $\pi$ , and call  $\pi$  a *derangement* if it does not have a fixed point.

In each round of AW, each player  $i$  plays a random permutation  $\pi^i$  of the locations  $[n] \setminus \{i\}$ , i.e., the first player plays a random permutation of the locations  $\{2, \dots, n\}$ , the second player plays a random permutation of the locations  $\{1\} \cup \{3, \dots, n\}$ , and we will be interested in the probability that they share a location, i.e., that  $\pi^1(j) = \pi^2(j)$  for some  $j \in [n-1]$ . As no meeting in location 1 can appear anyway, this is the same as the probability that two random permutations of  $\{2, \dots, n\}$  and  $\{3, \dots, n+1\}$  share a location which is, in turn, equal to the probability that two random permutations of  $\{1, \dots, n-1\}$  and  $\{2, \dots, n\}$  share a location. This probability is independent of the first permutation, so we may assume that it is the identity on the first  $n-1$  elements. So the analysis of the AW strategy naturally involves the probability that a permutation of the elements  $\{1+k, \dots, n+k\}$  is a derangement for  $k=1$ . As we will see, the analysis of FSD also involves that probability for larger values of  $k > 1$  because fixing the first position may remove further possibilities of meeting from the remaining round.

Thus, explicit and approximate values for this probability are needed. To obtain these, we first use a recursion dating back to Euler [1753] who defined a difference table with entries  $(d_n^k)$  for  $n \in \mathbb{N}_0$  and  $k \in \{0, \dots, n\}$  given by the following recurrence relation:

$$\begin{aligned} d_n^n &:= n! && \text{for all } n \in \mathbb{N}_0, \\ d_n^k &:= d_n^{k+1} - d_{n-1}^k && \text{for all } n \in \mathbb{N}, k \in \{1, \dots, n-1\}. \end{aligned}$$

Some initial rows and columns of this table are shown in Table 1. Euler's motivation was to calculate the number of derangements, and he proved that this number is equal to  $d_n^0$ . The following Lemma 1 is a slight generalization of this result, which is already implicit in Euler's work.

**Lemma 1.** *For all  $n \in \mathbb{N}_0$  and  $k \in \{0, \dots, n\}$ , the entry  $d_n^k$  is equal to the number of permutations  $\pi \in \mathcal{P}(\{1+k, \dots, n+k\})$  with  $\pi(i) \neq i$  for all  $i \in [n]$ .*

Lemma 1 was shown in a slightly different context by Feinsilver and McSorley [2011]. For the sake of completeness, we include a proof in our context and notation in Appendix A.1.

Feinsilver and McSorley also showed that

$$d_n^k = \sum_{j=k}^n (-1)^{n-j} \binom{n-k}{n-j} j! = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j)!,$$

which yields in particular  $d_n^0 = n! \sum_{j=0}^n \frac{(-1)^j}{j!}$ . Noting that  $\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} = 1/e$ , this shows that the number of derangements can be approximated by  $n!/e$ . Making the errors in this approximation explicit and using the recursive nature of the values  $d_n^k$ , we obtain the following bounds.

**Lemma 2.** *Let  $n \in \mathbb{N}$ . Then*

		$k$					
		0	1	2	3	4	5
$n$	0	1					
	1	0	1				
	2	1	1	2			
	3	2	3	4	6		
	4	9	11	14	18	24	
	5	44	53	64	78	96	120

Table 1: Euler's difference table

1.  $d_n^0 = \lfloor \frac{n!}{e} + \epsilon \rfloor$  for any  $\epsilon \in [\frac{1}{3}, \frac{1}{2}]$ ;
2.  $d_n^1 = \lfloor \frac{n!+(n-1)!}{e} + \epsilon \rfloor$  for any  $\epsilon \in [\frac{1}{3}, \frac{7}{8}]$ ;
3.  $d_n^2 = \lfloor \frac{n!+2(n-1)!+(n-2)!}{e} + \epsilon \rfloor$  for any  $\epsilon \in [\frac{1}{3}, \frac{7}{8}]$ .

We defer the proof to Appendix A.2. The calculations for the approximation of  $d_n^0$  were already given by Hassani [2004].

For  $n \in \mathbb{N}_0$  and  $k \in \{0, \dots, n\}$ , we denote by  $\hat{d}_n^k := d_n^k/n!$  the *fraction* of permutations in  $\mathcal{P}(\{1+k, \dots, n+k\})$  that do not have a fixed point.

### 3 Warm-Up: Correlating Consecutive Permutations

To gain intuition, and illustrate the key ideas behind our improvement over AW, let us first consider simplified strategies that treat the home location like any other location when moving. We will focus on meeting probabilities for this section.

Let Simplified Anderson-Weber with parameter  $\theta$ , or SAW( $\theta$ ) for short, be the  $n$ -strategy where

- (i)  $\chi_r^i = S$  with probability  $\theta$  and  $\chi_r^i = M$  with probability  $1 - \theta$  for each player  $i \in \{1, 2\}$  and round  $r \in \mathbb{N}$ , independently of all other rounds; and
- (ii)  $\pi_r^i$  is a permutation taken uniformly at random from  $\mathcal{P}([n])$  for each player  $i \in \{1, 2\}$  and round  $r \in \mathbb{N}$ , independently of all other rounds.

Let Simplified First-Same-or-Different with parameter  $\theta$ , or SFSD( $\theta$ ), be the  $n$ -strategy where

- (i)  $\chi_r^i = S$  with probability  $\theta$  and  $\chi_r^i = M$  with probability  $1 - \theta$  for each player  $i \in \{1, 2\}$  and round  $r \in \mathbb{N}$ , independently of all other rounds; and
- (ii) for each  $i \in \{1, 2\}$ ,  $\pi_r^i$  is a permutation taken uniformly at random from  $\mathcal{P}([n])$  for each round  $r \in \mathbb{N}$  with  $r/3 \notin \mathbb{N}$ , independently of all other rounds, and  $\pi_r^i$  is a permutation taken uniformly at random from the set

$$\begin{array}{ll} \{\pi \in \mathcal{P}([n]) : \pi(1) = \pi_{r-2}^i(1)\} & \text{if } \chi_{r-2}^i = \chi_{r-1}^i = \chi_r^i = M \text{ and } \pi_{r-2}^i(1) = \pi_{r-1}^i(1), \\ \{\pi \in \mathcal{P}([n]) : \pi(1) \notin \{\pi_{r-2}^i(1), \pi_{r-1}^i(1)\}\} & \text{if } \chi_{r-2}^i = \chi_{r-1}^i = \chi_r^i = M \text{ and } \pi_{r-2}^i(1) \neq \pi_{r-1}^i(1), \\ \mathcal{P}([n]) & \text{otherwise,} \end{array}$$

for all  $r \in \mathbb{N}$  with  $r/3 \in \mathbb{N}$ .

Focusing on rounds in which both players move,<sup>4</sup> players fail to meet if and only if the permutations they choose are derangements. Thus, under SAW, the probability that the players have not

<sup>4</sup>These are, in a sense, the *interesting* rounds when we are looking for an improvement: players do not necessarily meet but may be able to learn in order to better coordinate in subsequent rounds.

met within  $r \in \mathbb{N}$  moving rounds is  $(\hat{d}_n^0)^r$ . We will see that the probability is smaller under SFSD when  $r \geq 3$ .

### 3.1 Rendezvous with Visibility

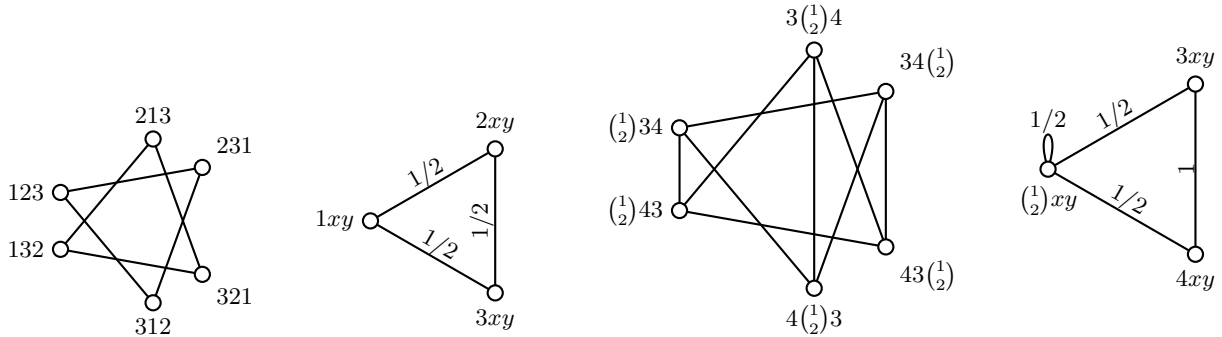
To understand this improvement, it is helpful to view SAW and SFSD as operating on the complement of the derangement graph on  $\mathcal{P}([n])$ . Vertices in this graph are permutations in  $\mathcal{P}([n])$ , and two different vertices are connected by an edge if they are *not* derangements. Rendezvous in a particular round occurs if the players choose the same vertex or two neighboring vertices.

This motivates a generalization of the rendezvous problem on a vertex-transitive *visibility graph*  $G = (V, E)$ , where players choose a vertex in each time step and meet if they have chosen vertices  $u$  and  $v$  such that  $u = v$  or  $\{u, v\} \in E$ . We obtain the original rendezvous problem by setting  $E = \emptyset$ . Recall that a graph  $G = (V, E)$  is *vertex-transitive* if for all  $u, v \in V$  there is a permutation  $f: V \rightarrow V$  with  $f(u) = v$  and  $(w, x) \in E$  if and only if  $(f(w), f(x)) \in E$ . Vertex-transitivity is a useful property for visibility graphs, since in a vertex-transitive graph all vertices look the same and players therefore cannot coordinate a priori on a subset of vertices. For a given visibility graph, a rendezvous strategy  $\sigma$ , as well as the random variables  $T_\sigma$  and  $t_\sigma$ , can be defined as before.<sup>5</sup> We further define  $\delta := \frac{1}{|V|^2} |\{u, v \in V : u \neq v, \{u, v\} \notin E\}|$ , and note that  $\delta$  is the probability that the players fail to meet in a single round when choosing vertices uniformly at random.

The round-based strategies we have introduced for the original rendezvous problem can be understood as strategies for rendezvous with visibility on the complement of the derangement graph, which is clearly vertex-transitive. SAW chooses a permutation uniformly at random in each moving round and thus corresponds to the *uniform strategy* UNI for rendezvous with visibility, which in every step visits a vertex chosen uniformly at random. SFSD, on the other hand, belongs to a family of strategies we will call  *$\ell$ -clique strategies*. Call a partition  $V_1, \dots, V_m$  of the set  $V$  a *clique partitioning* if  $V_i$  is a clique with  $|V_i| = q$  for all  $i \in [m]$ . Graphs that admit such a partitioning are also called (weakly)  $(m, q)$ -clique-partitioned [Erskine, Griggs, and Sirán, 2022]; for further properties of vertex-transitive clique-partitioned graphs, see, e.g., Dobson, Hujdurovic, Milanic, and Verret [2015]. It is an easy consequence of vertex transitivity that each vertex has the same number of edges to vertices in other cliques, i.e., there is  $\delta_{\text{out}}$  such that, for all  $i, j \in [m]$  and  $u \in V_i$ ,  $|\{v \in V_j : \{u, v\} \in E\}| = \delta_{\text{out}}$ .<sup>6</sup> Given a vertex-transitive  $(m, q)$ -clique-partitioned graph, and  $\ell \in \mathbb{N}$  with  $\ell \leq \min\{m - 1, q - 1\}$ , the  $\ell$ -clique strategy with parameter  $\mu \in [0, 1]$ , denoted by  $\text{CLI}_\ell(\mu)$ , operates in rounds of length  $\ell$ . For each round  $r \in \mathbb{N}$ , and independently of all other rounds, a player stays with probability  $\mu$  and moves with probability  $1 - \mu$ . The player then chooses for each step between  $(r - 1)\ell + 1$  and  $r\ell$  a vertex uniformly at random from a restricted subset of vertices; when staying she chooses, independently from the other steps, vertices from her *home clique*, i.e., the clique containing the location she visited at step 0; when moving she chooses from cliques not yet visited in this round.

<sup>5</sup>We again focus on  $t_\sigma$  and assume that players chose different locations and did not meet at time 0.

<sup>6</sup>To see this, note that each automorphism can be decomposed into a permutation of the cliques and  $m$  permutations of the vertices within the cliques.



(a) Graphs representing the permutations that our simplified strategies from Section 3 play for  $n = 3$  locations. The non-meeting probability when visiting two vertices taken uniformly at random in the original graph is  $\delta = 1/3$ , and when visiting two different vertices in the new graph is  $\frac{n}{n-1}\delta = \frac{1}{2}$ .

(b) Graphs representing the permutations that our strategies from Section 4 play for  $n = 4$  locations. We write  $\binom{1}{2}$  for the home location of the other player; two permutations meet only if they coincide at another location (3 or 4). In particular, a self-loop appears because players are not guaranteed to meet when they both visit permutations that start with  $\binom{1}{2}$ .

Figure 1: Examples of original and contracted derangement graphs. The original graphs contain edges between pairs of permutations that do not meet; the contracted graphs have edges between vertices that represent a set of permutations, and the label corresponds to the non-meeting probability when choosing one permutation from each endpoint uniformly at random.

### 3.2 Beating the Uniform Strategy

On an intuitive level,  $\ell$ -clique strategies mirror  $\ell$ -strategies like AW in a way that exploits the structure of the visibility graph: Players in staying rounds now stay at their home clique but move randomly within it; players in moving rounds pick both an unvisited clique and a location within it uniformly at random. The random choice within a clique is made for simplicity, to keep the meeting probability constant for all steps of a round under the condition of visiting distinct cliques. What enables this kind of strategy to achieve an improvement over the uniform strategy is that it shifts probability mass towards the event that both players visit the same clique when at least one of them moves, an event that guarantees rendezvous.

It is again helpful to look at the complement of the visibility graph. The uniform strategy has a non-meeting probability, after  $\ell$  steps, equal to  $\delta^\ell$ . On the other hand,  $\ell$ -clique strategies reduce the problem to a contracted graph where the  $n$  independent sets become vertices and players choose either the same vertex or  $\ell$  distinct vertices for the  $\ell$  steps of a round. When visiting two distinct vertices, the non-meeting probability is  $\frac{m}{m-1}\delta$ , because contracted vertices had no edges between them and contraction has thus not changed the average degree. Figure 1a shows the original and the contracted derangement graph for permutations of length 3. The trade-off is now clear: compared to the uniform strategy,  $\ell$ -clique strategies are played on a smaller graph—where visiting the same vertex is easier—but with lower visibility—making rendezvous harder when visiting distinct vertices. It turns out that lower visibility dominates when  $\ell = 1$ , but that the contraction pays off already when  $\ell = 2$ .

**Proposition 1.** *Let  $G = (V, E)$  be a vertex-transitive  $(m, q)$ -clique-partitioned graph, where  $m \geq 3$  and  $q \geq 3$ . Then,  $\mathbb{P}[t_{\text{CLI}_1(\mu)} \leq 1] \leq \mathbb{P}[t_{\text{UNI}} \leq 1]$  for every  $\mu \in [0, 1]$ , and  $\mathbb{P}[t_{\text{CLI}_2(1/m)} \leq r] > \mathbb{P}[t_{\text{UNI}} \leq r]$  for every  $r \in \mathbb{N}$  with  $r \geq 2$ .*

We prove this proposition in Appendix B.1 by showing that  $\mathbb{P}[t_{\text{CLI}_1(\mu)} \leq 1] \leq \delta$  for every  $\mu \in [0, 1]$

and that  $\mathbb{P}[t_{\text{CLI}_2(1/m)} \leq 2] > \delta^2$ . We determine the values of these expressions as a function of the parameter  $\mu$  by calculating the probabilities that the players do not visit the same clique in the first or first two steps, conditioning on the number of moving players, and applying the law of total probability.

Proposition 1 implies immediately that SFSD( $\theta$ ) improves over SAW( $\theta$ ) in terms of meeting probabilities after three rounds. Indeed, when moving for three consecutive rounds, SAW( $\theta$ ) takes all permutations uniformly at random, while SFSD( $\theta$ ) takes the second and third permutations according to  $\text{CLI}_2(1/m)$ , with  $m = n$  cliques of size  $q = (n - 1)!$  defined by fixing the first element of each permutation. Since these cliques have only size 2 when the number of locations is  $n = 3$ , and it is not possible to partition the associated visibility graph into  $m \geq 3$  cliques of size  $q \geq 3$ , the improvement applies for  $n \geq 4$ .

**Corollary 1.** *For every  $n \in \mathbb{N}$  with  $n \geq 4$ , every  $\theta \in [0, 1]$ , and every  $r \in \mathbb{N}$  with  $r \geq 3$ ,*

$$\mathbb{P}[t_{\text{SFSD}(\theta)} \leq rn] > \mathbb{P}[t_{\text{SAW}(\theta)} \leq rn].$$

SFSD does not improve over AW, even in terms of meeting probabilities. This is due to the fact that SFSD visits the home location even in moving rounds and thus uses rounds of length  $n$  compared to length  $n - 1$  for AW. We will resolve this issue by returning to FSD, and will also obtain an improvement of the meeting time. The analysis will become much more involved.

## 4 Beating the Anderson–Weber Strategy

We will now show our main result, that FSD has a strictly smaller expected meeting time than AW for any  $n \geq 4$ . The analysis relies heavily on an intermediate result that we have made plausible in Section 3: for  $n \geq 5$ , FSD has a higher probability of meeting than AW after  $r$  rounds for any  $r \geq 3$ . To see why this is useful and sketch the general proof strategy, we observe that for a strategy  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ , the expected meeting time can be written as

$$\mathbb{E}[t_\sigma] = \mathbb{E}[t_\sigma \mid \mathbb{R}_\sigma^{\leq 3}] \mathbb{P}[\mathbb{R}_\sigma^{\leq 3}] + (3(n - 1) + E[t_\sigma])(1 - \mathbb{P}[\mathbb{R}_\sigma^{\leq 3}]),$$

because both strategies restart after  $3(n - 1)$  steps. Therefore,

$$\mathbb{E}[t_\sigma] = \mathbb{E}[t_\sigma \mid \mathbb{R}_\sigma^{\leq 3}] + 3(n - 1) \left( \frac{1}{\mathbb{P}[\mathbb{R}_\sigma^{\leq 3}]} - 1 \right),$$

so that  $\mathbb{E}[t_\sigma]$  is decreasing in  $\mathbb{P}[\mathbb{R}_\sigma^{\leq 3}]$  and increasing in  $\mathbb{E}[t_\sigma \mid \mathbb{R}_\sigma^{\leq 3}]$ . We show in Section 4.1 that  $\mathbb{P}[\mathbb{R}_\sigma^{\leq 3}]$  is greater for FSD than for AW, and in Section 4.2 that  $\mathbb{E}[t_\sigma \mid \mathbb{R}_\sigma^{\leq 3}]$  is smaller for FSD than for AW.

Before we start the technical analysis, let us briefly explain the intuition behind our improvement over AW in light of the analysis in Section 3. Like its simplified version from Section 3, FSD correlates consecutive permutations in a way that emulates an Anderson–Weber-like strategy on the complement of the derangement graph. Specifically, for three consecutive permutations, it chooses the first two uniformly at random, and the third uniformly at random subject to the constraint that the three permutations must start with the same location or three distinct locations. However, to actually improve over AW, it will be necessary as in AW to use a home location in which a player stays in staying rounds and which can safely be excluded from moving rounds. The existence

of the home location considerably complicates the analysis, since now the permutations players choose have length  $n - 1$  and differ in one element. As a consequence the derangement graph is no longer symmetric; it is now a graph with the same set of vertices as the derangement graph on  $n - 1$  locations, but with additional edges corresponding to players not meeting when visiting each other's home location. This is illustrated in Figure 1b.

In the following sections, we focus our analysis on the first three rounds and exploit the fact that both of the strategies we analyze restart every  $3(n - 1)$  steps. We need some additional notation. Let  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ , and let  $\chi_{\sigma,r}^i \in \{\text{S}, \text{M}\}$  denote the (random) decision of player  $i \in \{1, 2\}$  in strategy  $\sigma$  to stay or move in the  $r$ th round. In a slight abuse of notation, we will write  $\chi_{\sigma}^i \in \{\text{S}, \text{M}\}^2$  or  $\chi_{\sigma}^i \in \{\text{S}, \text{M}\}^3$  for the decisions of player  $i \in \{1, 2\}$  in the first two or first three rounds, and compress such 2- or 3-dimensional vectors with a repeated component to a single letter with the length as superscript; for example, we will use  $\chi_{\sigma}^i = \text{M}^2$  to denote that player  $i$  moves in the first two rounds. For  $i \in \{1, 2\}$  and  $r \in [3]$ , we let  $\pi_{\sigma,r}^i$  denote the (random) permutation in  $\mathcal{P}([n] \setminus \{i\})$  that player  $i$  takes in round  $r$  under strategy  $\sigma$  if  $\chi_{\sigma}^i = \text{M}$ . We will write  $\text{R}_{\sigma}^r := [(r - 1)(n - 1) < t_{\sigma} \leq r(n - 1)]$  and  $\text{R}_{\sigma}^{\leq r} := [t_{\sigma} \leq r(n - 1)]$  for the events that under strategy  $\sigma$  the players meet for the first time *during round  $r$*  and *up to round  $r$* , respectively. A bar above  $\text{R}$  will be used to denote negation, so that in particular  $\bar{\text{R}}_{\sigma}^{\leq r}$  means that the players have *not* met up to round  $r$ . Finally, the following events will be useful for the analysis of  $\text{FSD}(\theta)$ :

$$\begin{aligned} A_1^i &:= [\pi_{\text{FSD}(\theta),1}^i(1) = \pi_{\text{FSD}(\theta),2}^i(1) = 3 - i], \\ A_2^i &:= [\pi_{\text{FSD}(\theta),1}^i(1) = \pi_{\text{FSD}(\theta),2}^i(1) \neq 3 - i], \\ A_3^i &:= [\pi_{\text{FSD}(\theta),1}^i(1) \neq \pi_{\text{FSD}(\theta),2}^i(1), 3 - i \in \{\pi_{\text{FSD}(\theta),1}^i(1), \pi_{\text{FSD}(\theta),2}^i(1), \pi_{\text{FSD}(\theta),3}^i(1)\}], \\ A_4^i &:= [\pi_{\text{FSD}(\theta),1}^i(1) \neq \pi_{\text{FSD}(\theta),2}^i(1), 3 - i \notin \{\pi_{\text{FSD}(\theta),1}^i(1), \pi_{\text{FSD}(\theta),2}^i(1), \pi_{\text{FSD}(\theta),3}^i(1)\}]. \end{aligned}$$

Events  $A_1^i$  and  $A_2^i$  occur if player  $i$  starts the three tours at the same location, and if this location is or is not the home location of the other player, respectively. Similarly,  $A_3^i$  and  $A_4^i$  occur if player  $i$  starts the three tours at different locations, and one of them or none of them is the home location of the other player, respectively.<sup>7</sup> We finally define  $w_k^i := \mathbb{P}[A_k^i \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \text{M}^3]$ , for  $i \in \{1, 2\}$  and  $k \in [4]$ , to be the probability of  $A_k^i$  under the condition that both players move in all of the first three rounds. These probabilities are independent of  $\theta$  and are given by the following lemma. The simple proof is deferred to Appendix C.1.

**Lemma 3.** *For  $n \in \mathbb{N}$  with  $n \geq 4$ , and  $i \in \{1, 2\}$ ,*

$$w_1^i = \frac{1}{(n - 1)^2}, \quad w_2^i = \frac{n - 2}{(n - 1)^2}, \quad w_3^i = \frac{3(n - 2)}{(n - 1)^2}, \quad w_4^i = \frac{(n - 2)(n - 4)}{(n - 1)^2}.$$

## 4.1 Meeting Probabilities

We begin by looking at meeting probabilities, and we will see that FSD provides an improvement over AW for  $n \geq 5$  but not for  $n = 4$ . The following is our result.

<sup>7</sup>We have defined the events using only  $\pi_{\text{FSD}(\theta),1}^i(1)$  and  $\pi_{\text{FSD}(\theta),2}^i(1)$  for brevity, but by definition of  $\text{FSD}(\theta)$ ,  $\pi_{\text{FSD}(\theta),3}^i(1)$  will be the same as these or different from both of these depending on whether they are the same or differ.

**Theorem 1.** For every  $n \in \mathbb{N}$  with  $n \geq 5$  and every  $\theta \in [0, 1)$ , it holds

$$\mathbb{P}[t_{\text{FSD}(\theta)} \leq 3(n-1)] \geq \mathbb{P}[t_{\text{AW}(\theta)} \leq 3(n-1)] + \frac{967(1-\theta)^6}{(n-1)^9}.$$

For  $n = 4$  and every  $\theta \in [0, 1)$ ,  $\mathbb{P}[t_{\text{FSD}(\theta)} \leq 3(n-1)] = \mathbb{P}[t_{\text{AW}(\theta)} \leq 3(n-1)]$ .

Since  $\text{FSD}(\theta)$  repeats every 3 rounds and coincides with  $\text{AW}(\theta)$  in all rounds that are not multiples of 3, we have for every  $n \in \mathbb{N}$  with  $n \geq 5$ ,  $\theta \in [0, 1)$ ,  $k \in \mathbb{N}$ , and  $r \in \{0, 1, 2\}$  that

$$\mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3k+r}] = \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3}]^k \mathbb{P}[\bar{\mathbf{R}}_{\text{AW}(\theta)}^1]^r < \mathbb{P}[\bar{\mathbf{R}}_{\text{AW}(\theta)}^{\leq 3}]^k \mathbb{P}[\bar{\mathbf{R}}_{\text{AW}(\theta)}^1]^r = \mathbb{P}[\bar{\mathbf{R}}_{\text{AW}(\theta)}^{3k+r}],$$

where the inequality holds by Theorem 1. We thus obtain the following corollary.

**Corollary 2.** For every  $n, r \in \mathbb{N}$  with  $n \geq 5$  and  $r \geq 3$ , and  $\theta \in [0, 1)$ ,

$$\mathbb{P}[t_{\text{FSD}(\theta)} \leq r(n-1)] > \mathbb{P}[t_{\text{AW}(\theta)} \leq r(n-1)].$$

One of the key ingredients of the proof of Theorem 1 is the following lemma, which gives an explicit expression for the non-meeting probability under our strategy up to time  $3(n-1)$ , under the condition that both players move for the first three rounds. We also give the exact value for  $n \in \{4, 5, 6, 7\}$ .

**Lemma 4.** For every  $n \in \mathbb{N}$  with  $n \geq 4$ , and  $\theta \in [0, 1)$ ,

$$\begin{aligned} \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] = \\ \frac{1}{(n-1)^4(n-3)} \left( (n-3)(\hat{d}_{n-2}^0)^3 + 6(n-2)(n-3)\hat{d}_{n-2}^0(\hat{d}_{n-2}^1)^2 \right. \\ \quad + 3(n-2)(n^2 - 7n + 13)\hat{d}_{n-2}^0(\hat{d}_{n-2}^2)^2 \\ \quad + 2(n-2)(n-3)^2(\hat{d}_{n-2}^1)^3 + 6(n-2)(n-3)^2(\hat{d}_{n-2}^1)^2\hat{d}_{n-2}^2 \\ \quad + 6(n-2)(n-4)(n^2 - 6n + 10)\hat{d}_{n-2}^1(\hat{d}_{n-2}^2)^2 \\ \quad \left. + (n^5 - 16n^4 + 103n^3 - 335n^2 + 551n - 362)(\hat{d}_{n-2}^2)^3 \right). \end{aligned}$$

In particular, this value equals  $\frac{1}{8}$  for  $n = 4$ ,  $\frac{5}{54}$  for  $n = 5$ ,  $\frac{439471}{5184000}$  for  $n = 6$ , and  $\frac{406417}{5184000}$  for  $n = 7$ .

Since  $\hat{d}_n^k$  converges to  $1/e$  as  $n \rightarrow \infty$  for any fixed  $k$ , this non-meeting probability approaches  $1/e^3$  as  $n \rightarrow \infty$ . This is the same as the corresponding non-meeting probability for AW, so FSD does not improve over AW in the limit.

We prove Lemma 4 in Appendix C.2. In the proof we further condition on the events  $A_k^1$  and  $A_\ell^2$  for  $k, \ell \in [4]$ , whose probabilities are given in Lemma 3. For each combination of these events, we then compute the non-meeting probabilities in the first three rounds by carefully analyzing the probabilities that (i) the respective first locations of the players coincide in some round, and that (ii) one of the subsequent permutations (of length  $n-2$ ) has a fixed point under the condition that the first locations do not coincide. Lemma 1 provides the key ingredient for the latter.

We are now ready to prove Theorem 1.

*Proof of Theorem 1.* Fix  $n \in \mathbb{N}$  with  $n \geq 4$  and  $\theta \in [0, 1)$ . For  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ ,

$$\mathbb{P}[\mathbf{R}_\sigma^{\leq 3}] = \sum_{i \in \{1, 2\}} \sum_{\tilde{\chi}^i \in \{\mathbf{S}, \mathbf{M}\}^3} \mathbb{P}[\mathbf{R}_\sigma^{\leq 3} \mid \chi_\sigma^1 = \tilde{\chi}^1, \chi_\sigma^2 = \tilde{\chi}^2] \mathbb{P}[\chi_\sigma^1 = \tilde{\chi}^1, \chi_\sigma^2 = \tilde{\chi}^2].$$

The terms of the sum on the right-hand side are equal for  $\sigma = \text{AW}(\theta)$  and for  $\sigma = \text{FSD}(\theta)$ , with the exception of the one corresponding to  $\tilde{\chi}^1 = \tilde{\chi}^2 = \mathbf{M}^3$ . Since  $\mathbb{P}[\chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3] = (1 - \theta)^6 > 0$  for  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ , and since  $\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3} \mid \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] = 1 - (\hat{d}_{n-1}^1)^3$  by Lemma 1, we have that

$$\begin{aligned} \mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3}] - \mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3}] &= (1 - \theta)^6 (\mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] - (1 - (\hat{d}_{n-1}^1)^3)) \\ &= (1 - \theta)^6 ((\hat{d}_{n-1}^1)^3 - \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]). \end{aligned}$$

The proof is then completed with the following lemma.

**Lemma 5.** For  $n = 4$ ,  $\mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] = (\hat{d}_{n-1}^1)^3$ . For  $n \in \mathbb{N}$  with  $n \geq 5$ ,

$$(\hat{d}_{n-1}^1)^3 - \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \geq \frac{967}{(n-1)^9}.$$

We prove this lemma in Appendix C.3. We compute the difference exactly for  $n \in \{4, 5, 6, 7\}$  using the expressions from Lemma 4, while for  $n \geq 8$  we bound it using Lemma 2.  $\square$

## 4.2 Expected Meeting Time

We will now show that FSD improves on AW not just in terms of meeting probability but also in terms of expected meeting time. The following is our main result.

**Theorem 2.** Let  $\theta \in [0, 1)$ . Then  $\mathbb{E}[t_{\text{FSD}(\theta)}] = \mathbb{E}[t_{\text{AW}(\theta)}] - \frac{8(1-\theta)^6}{81(8-(3\theta^2-2\theta+1)^3)}$  if  $n = 4$ , and  $\mathbb{E}[t_{\text{FSD}(\theta)}] \leq \mathbb{E}[t_{\text{AW}(\theta)}] - \frac{483(1-\theta)^6}{(n-1)^8}$  if  $n \in \mathbb{N}$  with  $n \geq 5$ .

By replacing  $\theta$  by the value that minimizes  $\mathbb{E}[t_{\text{AW}(\theta)}]$  we obtain explicit improvements over the previous best strategy. For  $n = 4$  the optimum value of  $\theta$  is approximately 0.322, which yields  $\mathbb{E}[t_{\text{AW}(\theta)}] - \mathbb{E}[t_{\text{FSD}(\theta)}] = \frac{8 \cdot 0.678^6}{81(8-(3 \cdot 0.322^2 - 2 \cdot 0.322 + 1)^3)} \approx 0.00125$ . For  $n = 5$  the optimum value of  $\theta$  is approximately 0.3012, which yields  $\mathbb{E}[t_{\text{AW}(\theta)}] - \mathbb{E}[t_{\text{FSD}(\theta)}] \geq 483 \cdot 0.6988^6 / 4^8 \approx 0.00086$ .

For the proof of Theorem 1 we conditioned on the respective first locations visited by a player in three consecutive rounds, and distinguished whether these locations were all the same or all different and whether they included the home location of the other player. This was enough to allow us to compute the probability of meeting within the three rounds. Now we will not only be concerned with the event that players meet within the three rounds, but also with the particular round and step within that round in which they meet for the first time. We deal with this added difficulty by exploiting that FSD and AW behave in the same way in all rounds that are not multiples of three, which allows us to focus on the expected meeting time *conditioned on both players moving in the first three rounds and meeting for the first time in the third round*. Specifically, we will show the following lemma.

**Lemma 6.** *Let  $\theta \in [0, 1)$ . Then*

$$\mathbb{E}[t_{\text{FSD}(\theta)} \mid \mathbf{R}_{\text{FSD}(\theta)}^3, \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] = \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] - \frac{8}{81}$$

if  $n = 4$ , and

$$\mathbb{E}[t_{\text{FSD}(\theta)} \mid \mathbf{R}_{\text{FSD}(\theta)}^3, \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] < \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3]$$

if  $n \in \mathbb{N}$  with  $n \geq 5$ .

Combining this lemma with Theorem 1 makes it rather intuitive that FSD beats AW in terms of meeting times. Indeed, FSD and AW have the same meeting probabilities for the first two rounds and the same meeting times conditioned on meeting within these rounds; Theorem 1 means that FSD has a larger meeting probability in the third round if both players move in the first three rounds, and Lemma 6 implies that FSD has a smaller expected meeting time conditioned on this event. Proving the theorem rigorously still requires some work, and we will do so at the end of the section. Most of the section will be concerned with the proof of Lemma 6.

In order to compute the expectations in Lemma 6 we will further condition on the respective first locations the players visit in the third round, which under FSD are *not* independent of not having met in previous rounds. We will explicitly compute these conditional expectations and the associated conditional probabilities to bound the difference between the expected meeting times of FSD and AW.

For  $\theta \in [0, 1)$  and  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ , we define the events

$$\begin{aligned} B_{\sigma,1} &:= [|\{i \in \{1, 2\} : \pi_{\sigma,3}^i(1) = 3 - i\}| = 2], \\ B_{\sigma,2} &:= [|\{i \in \{1, 2\} : \pi_{\sigma,3}^i(1) = 3 - i\}| = 1], \\ B_{\sigma,3} &:= [|\{i \in \{1, 2\} : \pi_{\sigma,3}^i(1) = 3 - i\}| = 0, \pi_{\sigma,3}^1(1) \neq \pi_{\sigma,3}^2(1)], \\ B_{\sigma,4} &:= [\pi_{\sigma,3}^1(1) = \pi_{\sigma,3}^2(1)], \end{aligned}$$

where we recall that  $\pi_{\sigma,r}^i$  denotes the permutation in  $\mathcal{P}([n] \setminus \{i\})$  used by player  $i \in \{1, 2\}$  in round  $r \in \mathbb{N}$  under strategy  $\sigma$ . Event  $B_{\sigma,1}$  thus means that in the third round both players start at the home location of the other player,  $B_{\sigma,2}$  that exactly one player starts at the home location of the other player,  $B_{\sigma,3}$  that none of the players start at the home location of the other player but they start at distinct locations, and  $B_{\sigma,4}$  that both players start at the same location, which is of course not the home location of either player. We will condition on  $\chi_{\sigma}^1 = \chi_{\sigma}^2 = \mathbf{M}^3$  whenever we refer to these events, so that the third permutation of a player will indeed be the permutation according to which the players tours in the third round.

For  $n \in \mathbb{N}$  with  $n \geq 2$  and  $b \in [n-1] \cup \{0\}$ , let  $\varphi_b(n)$  denote the expected index of the first fixed point of a permutation taken uniformly at random from  $\mathcal{P}(\{1+b, \dots, n+b\})$ , under the condition that it has at least one fixed point, i.e.,

$$\varphi_b(n) := \mathbb{E}[\min\{i \in [n] : \pi(i) = i\} \mid \{i \in [n] : \pi(i) = i\} \neq \emptyset],$$

where the expectation is taken over the choice of  $\pi$ . The following lemma provides an explicit expression for this expectation, and we will use it in the proofs of both Lemma 6 and Theorem 2.

**Lemma 7.** For every  $n \in \mathbb{N}$  with  $n \geq 2$  and  $b \in [n-1] \cup \{0\}$ ,

$$\varphi_b(n) = \frac{1}{1 - \hat{d}_n^b} \left( \frac{(n+1)^2}{n+1-b} (1 - \hat{d}_{n+1}^b) - (n+1) \hat{d}_n^b \right).$$

We prove the lemma in Appendix D.1. The proof uses a recursive formula for the probability that a permutation taken uniformly at random from  $\mathcal{P}(\{1+b, \dots, n+b\})$  has exactly  $k$  fixed points, where  $k \in [n-b]$ . This generalizes a technique used by Anderson and Weber [1990] to prove the special case where  $b = 0$ .

The proof of Lemma 6 uses two main ingredients, which we state as lemmas below. The first of these applies Lemma 7 to obtain explicit expressions for the expected meeting time under  $\text{AW}(\theta)$  or  $\text{FSD}(\theta)$ , under the condition (i) that both players move for the first three rounds, (ii) that they meet for the first time in the third round, and (iii) whether the respective first locations they visit in the third round are the same or different and whether they include the home location of the other player. The proof of this lemma is deferred to Appendix D.2.

**Lemma 8.** For every  $n \in \mathbb{N}$  with  $n \geq 4$ ,  $\theta \in [0, 1)$ , and  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ ,

$$\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,1}] = 2(n-1) + 1 + \frac{1}{1 - \hat{d}_{n-2}^0} (n-1)(1 - \hat{d}_{n-1}^0 - \hat{d}_{n-2}^0),$$

$$\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,2}] = 2(n-1) + 1 + \frac{1}{1 - \hat{d}_{n-2}^1} \left( \frac{(n-1)^2}{n-2} (1 - \hat{d}_{n-1}^1) - (n-1) \hat{d}_{n-2}^1 \right),$$

$$\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,4}] = 2(n-1) + 1.$$

For every  $n \in \mathbb{N}$  with  $n \geq 5$ ,  $\theta \in [0, 1)$ , and  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ ,

$$\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,3}] = 2(n-1) + 1 + \frac{1}{1 - \hat{d}_{n-2}^2} \left( \frac{(n-1)^2}{n-3} (1 - \hat{d}_{n-1}^2) - (n-1) \hat{d}_{n-2}^2 \right).$$

We note that the last expectation is not well defined when  $n = 4$  since  $\mathbf{R}_\sigma^3$  and  $B_{\sigma,3}$  are disjoint events in this case.

The second main ingredient, which will allow us to use the conditional expectations in Lemma 8, are expressions for the probabilities of the events  $B_{\sigma,k}$ , for  $k \in [4]$ , under the condition that both players move for the first three rounds and meet for the first time in the third round. Computing these probabilities is straightforward for  $\text{AW}$ , but fairly difficult for  $\text{FSD}(\theta)$  due to correlations among rounds.

The following lemma provides explicit expressions for the probability, under  $\text{FSD}(\theta)$  and the condition that both players move in the first three rounds, that (i) the players do not meet in the first two rounds and (ii) event  $B_{\text{FSD}(\theta),k}$  takes place for  $k \in [3]$ . The probability of the event  $B_{\text{FSD}(\theta),k}$  under the same condition can then be computed via Bayes' rule.

**Lemma 9.** For every  $n \in \mathbb{N}$  with  $n \geq 4$  and  $\theta \in [0, 1)$ ,

$$\begin{aligned} & \mathbb{P}[B_{\text{FSD}(\theta),1}, \bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 2} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ &= \frac{1}{(n-1)^4} \left( (\hat{d}_{n-2}^0)^2 + 2(n-2)(\hat{d}_{n-2}^1)^2 + \frac{(n-2)(n^2 - 7n + 13)}{n-3} (\hat{d}_{n-2}^2)^2 \right), \end{aligned}$$

$$\begin{aligned} & \mathbb{P}[B_{\text{FSD}(\theta),2}, \bar{R}_{\text{FSD}(\theta)}^{\leq 2} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = M^3] \\ &= \frac{2(n-2)}{(n-1)^4} \left( 2\hat{d}_{n-2}^0 \hat{d}_{n-2}^1 + (n-3)(\hat{d}_{n-2}^1)^2 + 2(n-3)\hat{d}_{n-2}^1 \hat{d}_{n-2}^2 + \frac{(n-4)(n^2-6n+10)}{n-3} (\hat{d}_{n-2}^2)^2 \right), \end{aligned}$$

$$\begin{aligned} & \mathbb{P}[B_{\text{FSD}(\theta),3}, \bar{R}_{\text{FSD}(\theta)}^{\leq 2} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = M^3] \\ &= \frac{(n-2)}{(n-1)^4} \left( \frac{2(n^2-7n+13)}{n-3} \hat{d}_{n-2}^0 \hat{d}_{n-2}^2 + 2(n-3)(\hat{d}_{n-2}^1)^2 + \frac{4(n-4)(n^2-6n+10)}{n-3} \hat{d}_{n-2}^1 \hat{d}_{n-2}^2 \right. \\ & \quad \left. + \frac{n^4-14n^3+75n^2-185n+181}{n-3} (\hat{d}_{n-2}^2)^2 \right). \end{aligned}$$

The long proof is deferred to Appendix D.3. It fixes  $j \in [3]$  and uses the law of total probability to further condition on the events  $A_k^1$  and  $A_\ell^2$  for  $k, \ell \in [4]$  and write, for each of them, the conditional probabilities as a product of the form

$$\begin{aligned} & \mathbb{P}[\bar{R}_{\text{FSD}(\theta)}^{\leq 2} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = M^3, B_{\text{FSD}(\theta),j}, A_k^1, A_\ell^2] \\ & \cdot \mathbb{P}[B_{\text{FSD}(\theta),j} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = M^3, A_k^1, A_\ell^2] \cdot \mathbb{P}[A_k^1, A_\ell^2 \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = M^3]. \end{aligned}$$

The first of these probabilities can be computed by analyzing the non-meeting probabilities in the first two rounds, under different conditions on the first locations of each of the first three rounds. The second probability can be obtained by understanding how the events  $A_k^1$  and  $A_\ell^2$ , which concern the first locations of all three rounds, affect the probabilities of the events  $B_{\text{FSD}(\theta),j}$ , which concern the first locations of the third round. The third probability is given by Lemma 3.

We now proceed with the proof of Lemma 6, which consists of two main steps. First, we use Lemma 8 to write the conditional expectations in the statement of the lemma in terms of (i) fractions of (shifted) derangements and (ii) probabilities of the events  $B_{\sigma,k}$  under the condition of moving in all three rounds and meeting for the first time in the third round. This allows us to further write the difference between the conditional expectations under FSD and AW as a function of the differences between the conditional probabilities. In the second step, we use Lemma 9 to compute and analyze these differences. Specifically, we show that the conditional probabilities of the events  $B_{\sigma,1}$  and  $B_{\sigma,4}$  are strictly greater under FSD than under AW, while the opposite relationship holds for the events  $B_{\sigma,2}$  and  $B_{\sigma,3}$ . Intuitively this is true because, conditional on moving and meeting for the first time in the third round, FSD has a larger bias towards (i) meeting in the first step of the third round, which leads to the best-possible meeting time in this round, and (ii) both players visiting the home location of the other player in the first step of the third round, which leads to an earlier meeting time later in the round compared to a situation where players visit distinct locations that are not the other player's home location.

*Proof of Lemma 6.* Fix  $n \in \mathbb{N}$  with  $n \geq 4$  and  $\theta \in [0, 1)$ . We first expand the conditional expectations in the statement by conditioning on the first location the players visit in the third round, captured by the events  $\{B_{\sigma,k}\}_{k \in [4]}$ . Letting

$$K := \begin{cases} [4] & \text{if } n \geq 5, \\ \{1, 2, 4\} & \text{if } n = 4 \end{cases}$$

denote the set of indices  $k$  such that the event  $B_{\sigma,k}$  is compatible with meeting in the third round we have, for  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ , that

$$\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3] = \sum_{k \in K} \mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,k}] \mathbb{P}[B_{\sigma,k} \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3]. \quad (1)$$

Explicit expressions for the conditional expectations on the right-hand side are given by Lemma 8. For the probabilities we use Bayes' Rule to see that for each  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$  and  $k \in K$ ,

$$\begin{aligned} \mathbb{P}[B_{\sigma,k} \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3] &= \mathbb{P}[B_{\sigma,k} \mid \mathbf{R}_\sigma^{\leq 3}, \bar{\mathbf{R}}_\sigma^{\leq 2}, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3] \\ &= \frac{\mathbb{P}[\mathbf{R}_\sigma^{\leq 3} \mid B_{\sigma,k}, \bar{\mathbf{R}}_\sigma^{\leq 2}, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3] \mathbb{P}[B_{\sigma,k} \mid \bar{\mathbf{R}}_\sigma^{\leq 2}, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3]}{\mathbb{P}[\mathbf{R}_\sigma^{\leq 3} \mid \bar{\mathbf{R}}_\sigma^{\leq 2}, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3]}, \end{aligned}$$

and observe that

$$\mathbb{P}[\mathbf{R}_\sigma^{\leq 3} \mid B_{\sigma,k}, \bar{\mathbf{R}}_\sigma^{\leq 2}, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3] = \begin{cases} 1 - \hat{d}_{n-2}^{k-1} & \text{if } k \in [3] \\ 1 & \text{if } k = 4. \end{cases}$$

To see the latter, consider a situation where both players move for three rounds and do not meet in the first two. If we condition on  $B_{\sigma,4}$ , they meet in the first step of the third round; otherwise, they meet in the third round if the respective permutations they choose for the remaining  $n-2$  steps are *not* derangements. If we condition on  $B_{\sigma,1}$ , both of these permutations are chosen uniformly at random from  $\mathcal{P}(\{3, \dots, n\})$ , so by Lemma 1 they are *not* derangements with probability  $1 - \hat{d}_{n-2}^0$ . If we condition on  $B_{\sigma,2}$ , and assuming w.l.o.g. that  $\pi_{\sigma,3}^1(1) = 2$  and  $\pi_{\sigma,3}^2(1) = j \neq 1$ , the permutation of player 1 is chosen uniformly at random from  $\mathcal{P}(\{3, \dots, n\})$  and that of player 2 from  $\mathcal{P}([n] \setminus \{2, j\})$ , so by Lemma 1 they are *not* derangements with probability  $1 - \hat{d}_{n-2}^1$ . If we condition on  $B_{\sigma,3}$ , and assuming w.l.o.g. that  $\pi_{\sigma,3}^1(1) = j \neq 2$  and  $\pi_{\sigma,3}^2(1) = k \notin \{1, j\}$ , the permutation of player 1 is chosen uniformly at random from  $\mathcal{P}([n] \setminus \{1, j\})$  and that of player 2 from  $\mathcal{P}([n] \setminus \{2, k\})$ , so by Lemma 1 they are *not* derangements with probability  $1 - \hat{d}_{n-2}^2$ . Replacing expressions in (1) accordingly, we conclude that, for  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ ,

$$\begin{aligned} (\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3] - 2(n-1) - 1) \mathbb{P}[\mathbf{R}_\sigma^{\leq 3} \mid \bar{\mathbf{R}}_\sigma^{\leq 2}, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3] &= \\ (n-1)(1 - \hat{d}_{n-1}^0 - \hat{d}_{n-2}^0) \mathbb{P}[B_{\sigma,1} \mid \bar{\mathbf{R}}_\sigma^{\leq 2}, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3] & \\ + \left( \frac{(n-1)^2}{n-2} (1 - \hat{d}_{n-1}^1) - (n-1) \hat{d}_{n-2}^1 \right) \mathbb{P}[B_{\sigma,2} \mid \bar{\mathbf{R}}_\sigma^{\leq 2}, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3] & \\ + \mathbb{1}_{n \geq 5} \left( \frac{(n-1)^2}{n-3} (1 - \hat{d}_{n-1}^2) - (n-1) \hat{d}_{n-2}^2 \right) \mathbb{P}[B_{\sigma,3} \mid \bar{\mathbf{R}}_\sigma^{\leq 2}, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3]. & \quad (2) \end{aligned}$$

For  $k \in [3]$ , let

$$\begin{aligned} \Delta_k &:= \mathbb{P}[B_{\text{AW}(\theta),k} \mid \bar{\mathbf{R}}_{\text{AW}(\theta)}^{\leq 2}(\theta), \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] \\ &\quad - \mathbb{P}[B_{\text{FSD}(\theta),k} \mid \bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 2}(\theta), \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]. \end{aligned}$$

The following lemma, establishing bounds on the differences  $\{\Delta_k\}_{k \in [3]}$ , is shown in Appendix D.4 using Lemma 9.

**Lemma 10.** *If  $n = 4$ , then  $\Delta_2 = -2\Delta_1 = 8/81$ . If  $n \geq 5$ , then  $\Delta_2 = -2\Delta_1 > 0$  and  $\Delta_3 > 0$ .*

We now consider the cases where  $n = 4$  and  $n \geq 5$  in turn. If  $n = 4$ , then by Theorem 1,

$$\begin{aligned} \mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3} \mid \bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 2}, \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] &= \mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3} \mid \bar{\mathbf{R}}_{\text{AW}(\theta)}^{\leq 2}, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] \\ &= 1 - \hat{d}_3^1 = \frac{1}{2}. \end{aligned} \quad (3)$$

Thus

$$\begin{aligned} &\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] - \mathbb{E}[t_{\text{FSD}(\theta)} \mid \mathbf{R}_{\text{FSD}(\theta)}^3, \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ &= 2 \left( \left( \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] - 2(n-1) - 1 \right) \right. \\ &\quad \cdot \mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3} \mid \bar{\mathbf{R}}_{\text{AW}(\theta)}^{\leq 2}, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] \\ &\quad \left. - \left( \mathbb{E}[t_{\text{FSD}(\theta)} \mid \mathbf{R}_{\text{FSD}(\theta)}^3, \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] - 2(n-1) - 1 \right) \right. \\ &\quad \left. \cdot \mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3} \mid \bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 2}, \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \right) \\ &= 2 \left( (n-1)(1 - \hat{d}_{n-1}^0 - \hat{d}_{n-2}^0) \Delta_1 + \left( \frac{(n-1)^2}{n-2} (1 - \hat{d}_{n-1}^1) - (n-1) \hat{d}_{n-2}^1 \right) \Delta_2 \right) \\ &= \frac{8(n-1)}{81} \left( \frac{2(n-1)}{n-2} (1 - \hat{d}_{n-1}^1) - 2\hat{d}_{n-2}^1 - (1 - \hat{d}_{n-1}^0 - \hat{d}_{n-2}^0) \right) \\ &= \frac{8}{27} \left( \frac{6}{2} \left( 1 - \frac{3}{6} \right) - 2 \cdot \frac{1}{2} - 1 + \frac{2}{6} + \frac{1}{2} \right) = \frac{8}{27} \cdot \frac{1}{3} = \frac{8}{81}, \end{aligned}$$

where the first equality follows from (3), the second from (2), the third from Lemma 10, and the last two from simple calculations. This completes the proof for  $n = 4$ .

If  $n \geq 5$ , we claim that

$$\begin{aligned} &\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] - \mathbb{E}[t_{\text{FSD}(\theta)} \mid \mathbf{R}_{\text{FSD}(\theta)}^3, \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ &\geq \left( \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] - 2(n-1) - 1 \right) \\ &\quad \cdot \mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3} \mid \bar{\mathbf{R}}_{\text{AW}(\theta)}^{\leq 2}, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] \\ &\quad - \left( \mathbb{E}[t_{\text{FSD}(\theta)} \mid \mathbf{R}_{\text{FSD}(\theta)}^3, \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] - 2(n-1) - 1 \right) \\ &\quad \cdot \mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3} \mid \bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 2}, \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ &= (n-1)(1 - \hat{d}_{n-1}^0 - \hat{d}_{n-2}^0) \Delta_1 + \left( \frac{(n-1)^2}{n-2} (1 - \hat{d}_{n-1}^1) - (n-1) \hat{d}_{n-2}^1 \right) \Delta_2 \\ &\quad + \left( \frac{(n-1)^2}{n-3} (1 - \hat{d}_{n-1}^2) - (n-1) \hat{d}_{n-2}^2 \right) \Delta_3. \\ &> \frac{n-1}{2} \left( \frac{2(n-1)}{n-2} (1 - \hat{d}_{n-1}^1) - 2\hat{d}_{n-2}^1 - (1 - \hat{d}_{n-1}^0 - \hat{d}_{n-2}^0) \right) \Delta_2. \end{aligned} \quad (4)$$

Indeed, the first inequality follows from Theorem 1, which implies that

$$\mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3} \mid \bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 2}, \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] > \mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3} \mid \bar{\mathbf{R}}_{\text{AW}(\theta)}^{\leq 2}, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3],$$

the equality from (2), and the last inequality from Lemma 10. Since  $\Delta_2 > 0$  from Lemma 10, the expression on the right-hand side of (4) is non-negative if the expression in parentheses is

non-negative. This is easy to check when  $n = 5$ , where the expression in parentheses evaluates to

$$\frac{8}{3} \left(1 - \frac{11}{24}\right) - 2 \cdot \frac{3}{6} - 1 + \frac{9}{24} + \frac{2}{6} = \frac{11}{72} > 0.$$

If  $n \geq 6$ , it can be shown by observing that

$$\begin{aligned} & \frac{2(n-1)}{n-2} (1 - \hat{d}_{n-1}^1) - 2\hat{d}_{n-2}^1 - (1 - \hat{d}_{n-1}^0 - \hat{d}_{n-2}^0) \\ & \geq \frac{2(n-1)}{n-2} \left(1 - \left(\frac{1}{e} + \frac{1}{(n-1)e} + \frac{1}{3(n-1)!}\right)\right) - 2\left(\frac{1}{e} + \frac{1}{(n-2)e} + \frac{1}{3(n-2)!}\right) \\ & \quad - 1 + \left(\frac{1}{e} - \frac{1}{2(n-1)!}\right) + \left(\frac{1}{e} - \frac{1}{2(n-2)!}\right) \\ & = 1 - \frac{2}{e} \left(1 + \frac{3-e}{n-2}\right) - \frac{7}{6(n-2)!} - \frac{2}{3(n-2)(n-2)!} - \frac{1}{2(n-1)!}, \end{aligned}$$

where we have use the bounds from Lemma 2 to obtain the inequality. The last expression is approximately equal to 0.1527 for  $n = 6$  and trivially increasing in  $n$ , which completes the proof.  $\square$

We now proceed with the proof of Theorem 2. Our goal will be to move from expected meeting times under strategy  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ , conditioned on the events  $\mathbf{R}_\sigma^3$  and  $\chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3$ , to unconditional expected meeting times. To this end, we first exploit the fact that both AW and FSD restart every  $3(n-1)$  steps to write the unconditional expectations in terms of conditional expectations of the form  $\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^{\leq 2}]$ , which are the same for both strategies, and  $\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3]$ , whose difference we bound in a second step. To express these conditional expectations in terms of the expectations that also condition on events of the type  $\chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3$ , we proceed differently for  $n = 4$  and for  $n \geq 5$ . For  $n = 4$  the meeting probabilities within each round are the same under both strategies due to Theorem 1, which simplifies the analysis. For  $n \geq 5$  we show that, under the condition of meeting in the third round, (i) the expected meeting time is lower when both players move in the first three rounds and (ii) this event has a higher probability under FSD( $\theta$ ) than under AW( $\theta$ ) due to Theorem 1.

*Proof of Theorem 2.* We fix  $\theta \in [0, 1)$  throughout the proof and make two initial observations. First, for  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ ,

$$\mathbb{E}[t_\sigma] = \mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^{\leq 2}] \mathbb{P}[\mathbf{R}_\sigma^{\leq 2}] + \mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3] \mathbb{P}[\mathbf{R}_\sigma^3] + (3(n-1) + \mathbb{E}[t_\sigma]) (1 - \mathbb{P}[\mathbf{R}_\sigma^{\leq 3}]),$$

because both strategies restart after  $3(n-1)$  steps and the events  $\mathbf{R}_\sigma^{\leq 2}$  and  $\mathbf{R}_\sigma^3$  are contained in the event  $\mathbf{R}_\sigma^{\leq 3}$ . Therefore

$$\begin{aligned} \mathbb{E}[t_\sigma] &= \frac{\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^{\leq 2}] \mathbb{P}[\mathbf{R}_\sigma^{\leq 2}] + \mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3] \mathbb{P}[\mathbf{R}_\sigma^3]}{\mathbb{P}[\mathbf{R}_\sigma^{\leq 3}]} + 3(n-1) \left(\frac{1}{\mathbb{P}[\mathbf{R}_\sigma^{\leq 3}]} - 1\right) \\ &= \mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^{\leq 2}] + (\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3] - \mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^{\leq 2}]) \mathbb{P}[\mathbf{R}_\sigma^3 \mid \mathbf{R}_\sigma^{\leq 3}] + 3(n-1) \left(\frac{1}{\mathbb{P}[\mathbf{R}_\sigma^{\leq 3}]} - 1\right). \quad (5) \end{aligned}$$

Second,

$$\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3] - \mathbb{E}[t_{\text{FSD}(\theta)} \mid \mathbf{R}_{\text{FSD}(\theta)}^3]$$

$$\begin{aligned}
&= \mathbb{E}[t_{AW(\theta)} \mid \mathbf{R}_{AW(\theta)}^3, \neg(\chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3)] \mathbb{P}[\neg(\chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3) \mid \mathbf{R}_{AW(\theta)}^3] \\
&\quad - \mathbb{E}[t_{FSD(\theta)} \mid \mathbf{R}_{FSD(\theta)}^3, \neg(\chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3)] \mathbb{P}[\neg(\chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3) \mid \mathbf{R}_{FSD(\theta)}^3] \\
&\quad + \mathbb{E}[t_{AW(\theta)} \mid \mathbf{R}_{AW(\theta)}^3, \chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3] \mathbb{P}[\chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3 \mid \mathbf{R}_{AW(\theta)}^3] \\
&\quad - \mathbb{E}[t_{FSD(\theta)} \mid \mathbf{R}_{FSD(\theta)}^3, \chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3] \mathbb{P}[\chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3 \mid \mathbf{R}_{FSD(\theta)}^3] \\
&= \mathbb{E}[t_{AW(\theta)} \mid \mathbf{R}_{AW(\theta)}^3, \neg(\chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3)] \\
&\quad \cdot (\mathbb{P}[\chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3 \mid \mathbf{R}_{FSD(\theta)}^3] - \mathbb{P}[\chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3 \mid \mathbf{R}_{AW(\theta)}^3]) \\
&\quad + \mathbb{E}[t_{AW(\theta)} \mid \mathbf{R}_{AW(\theta)}^3, \chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3] \mathbb{P}[\chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3 \mid \mathbf{R}_{AW(\theta)}^3] \\
&\quad - \mathbb{E}[t_{FSD(\theta)} \mid \mathbf{R}_{FSD(\theta)}^3, \chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3] \mathbb{P}[\chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3 \mid \mathbf{R}_{FSD(\theta)}^3], \tag{6}
\end{aligned}$$

where we have used that conditioned on meeting in the third round, the expected meeting times under both strategies are the same unless players move for all three rounds.<sup>8</sup>

We now consider the cases where  $n = 4$  and  $n \geq 5$  in turn. If  $n = 4$ , we claim that

$$\begin{aligned}
&\mathbb{E}[t_{AW(\theta)}] - \mathbb{E}[t_{FSD(\theta)}] \\
&= (\mathbb{E}[t_{AW(\theta)} \mid \mathbf{R}_{AW(\theta)}^3] - \mathbb{E}[t_{FSD(\theta)} \mid \mathbf{R}_{FSD(\theta)}^3]) \mathbb{P}[\mathbf{R}_{AW(\theta)}^3 \mid \mathbf{R}_{AW(\theta)}^{\leq 3}] \\
&= (\mathbb{E}[t_{AW(\theta)} \mid \mathbf{R}_{AW(\theta)}^3, \chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3] - \mathbb{E}[t_{FSD(\theta)} \mid \mathbf{R}_{FSD(\theta)}^3, \chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3]) \\
&\quad \cdot \mathbb{P}[\chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3 \mid \mathbf{R}_{AW(\theta)}^3] \mathbb{P}[\mathbf{R}_{AW(\theta)}^3 \mid \mathbf{R}_{AW(\theta)}^{\leq 3}] \\
&= \frac{8}{81} \cdot \frac{\mathbb{P}[\mathbf{R}_{AW(\theta)}^3 \mid \chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3] \mathbb{P}[\chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3]}{\mathbb{P}[\mathbf{R}_{AW(\theta)}^{\leq 3}]} \\
&= \frac{8}{81} \cdot \frac{(1 - \theta)^6 (\hat{d}_3^1)^2 (1 - \hat{d}_3^1)}{\mathbb{P}[\mathbf{R}_{AW(\theta)}^{\leq 3}]} = \frac{(1 - \theta)^6}{81 \cdot \mathbb{P}[\mathbf{R}_{AW(\theta)}^{\leq 3}]} \tag{7}
\end{aligned}$$

Indeed, the first equality follows from (5) and the fact that, by definition of the strategies and Theorem 1,

$$\begin{aligned}
\mathbb{E}[t_{AW(\theta)} \mid \mathbf{R}_{AW(\theta)}^{\leq 2}] &= \mathbb{E}[t_{FSD(\theta)} \mid \mathbf{R}_{FSD(\theta)}^{\leq 2}], \\
\mathbb{P}[\mathbf{R}_{AW(\theta)}^3] &= \mathbb{P}[\mathbf{R}_{FSD(\theta)}^3], \text{ and } \mathbb{P}[\mathbf{R}_{AW(\theta)}^{\leq 3}] = \mathbb{P}[\mathbf{R}_{FSD(\theta)}^{\leq 3}].
\end{aligned}$$

The second equality follows from (6) and the fact that, by definition of the strategies and Theorem 1,

$$\begin{aligned}
&\mathbb{P}[\chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3 \mid \mathbf{R}_{AW(\theta)}^3] \\
&= \frac{\mathbb{P}[\mathbf{R}_{AW(\theta)}^3 \mid \chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3] \mathbb{P}[\chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3]}{\mathbb{P}[\mathbf{R}_{AW(\theta)}^3]} \\
&= \frac{\mathbb{P}[\mathbf{R}_{FSD(\theta)}^3 \mid \chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3] \mathbb{P}[\chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3]}{\mathbb{P}[\mathbf{R}_{FSD(\theta)}^3]} \\
&= \mathbb{P}[\chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3 \mid \mathbf{R}_{FSD(\theta)}^3].
\end{aligned}$$

---

<sup>8</sup>This might not seem entirely obvious when one player moves in the first three rounds and the other one moves in the first two and stays in the last one. In this case, it holds because conditioning on both players moving and not meeting in the first two rounds does not affect the probability of visiting the other's home location at the beginning of the third round. This can be easily seen by applying Bayes' rule and observing that the event of a player visiting the other's home location in the first step of a round does not affect the meeting probability within that round.

The third equality holds by Lemma 6 and Bayes' rule, the fourth by definition of  $\text{AW}(\theta)$  and Lemma 1, and the last one because  $\hat{d}_3^1 = 1/2$ . Since

$$\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3}] = 1 - \mathbb{P}[\bar{\mathbf{R}}_{\text{AW}(\theta)}^1]^3 = 1 - (\theta^2 + (1-\theta)^2 \hat{d}_3^1)^3 = 1 - \frac{1}{8}(2\theta^2 + (1-\theta)^2)^3 = 1 - \frac{1}{8}(3\theta^2 - 2\theta + 1)^3,$$

we conclude from (7) that  $\mathbb{E}[t_{\text{AW}(\theta)}] - \mathbb{E}[t_{\text{FSD}(\theta)}] = \frac{8(1-\theta)^6}{81(8-(3\theta^2-2\theta+1)^3)}$ . This completes the proof for  $n = 4$ .

For  $n \geq 5$ , we claim that

$$\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3] \geq \mathbb{E}[t_{\text{FSD}(\theta)} \mid \mathbf{R}_{\text{FSD}(\theta)}^3]. \quad (8)$$

Indeed, by (6) and Lemma 6,

$$\begin{aligned} & \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3] - \mathbb{E}[t_{\text{FSD}(\theta)} \mid \mathbf{R}_{\text{FSD}(\theta)}^3] \\ & \geq (\mathbb{P}[\chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3 \mid \mathbf{R}_{\text{FSD}(\theta)}^3]) - \mathbb{P}[\chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3 \mid \mathbf{R}_{\text{AW}(\theta)}^3]) \\ & \quad \cdot (\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \neg(\chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3)] - \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3]), \end{aligned}$$

and we will see that both terms in parentheses on the right-hand side are non-negative. For the first term this holds because

$$\begin{aligned} & \mathbb{P}[\chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3 \mid \mathbf{R}_{\text{FSD}(\theta)}^3] - \mathbb{P}[\chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3 \mid \mathbf{R}_{\text{AW}(\theta)}^3] \\ & = \mathbb{P}[\neg(\chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3) \mid \mathbf{R}_{\text{AW}(\theta)}^3] - \mathbb{P}[\neg(\chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3) \mid \mathbf{R}_{\text{FSD}(\theta)}^3]) \\ & = \mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^3 \mid \neg(\chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3)] \mathbb{P}[\neg(\chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3)] \left( \frac{1}{\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^3]} - \frac{1}{\mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^3]} \right) \\ & \geq 0, \end{aligned}$$

where the second equality follows from Bayes' rule and the fact that the probabilities  $\mathbb{P}[\mathbf{R}_\sigma^3 \mid \neg(\chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3)]$  and  $\mathbb{P}[\neg(\chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3)]$  are identical for  $\sigma = \text{AW}(\theta)$  and  $\sigma = \text{FSD}(\theta)$ , and the inequality from Theorem 1 and the fact that  $\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 2}] = \mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 2}]$ . For the second term we note that for an event  $D \in \{[\chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3], [\neg(\chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3)]\}$ ,

$$\begin{aligned} & \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, D] \\ & = \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta),3}^1 = \chi_{\text{AW}(\theta),3}^2 = \mathbf{M}] \mathbb{P}[\chi_{\text{AW}(\theta),3}^1 = \chi_{\text{AW}(\theta),3}^2 = \mathbf{M} \mid \mathbf{R}_{\text{AW}(\theta)}^3, D] \\ & \quad + \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta),3}^1 \neq \chi_{\text{AW}(\theta),3}^2] \mathbb{P}[\chi_{\text{AW}(\theta),3}^1 \neq \chi_{\text{AW}(\theta),3}^2 \mid \mathbf{R}_{\text{AW}(\theta)}^3, D], \end{aligned}$$

because  $\mathbf{R}_{\text{AW}(\theta)}^3$  and  $\chi_{\text{AW}(\theta),3}^1 = \chi_{\text{AW}(\theta),3}^2 = \mathbf{S}$  are disjoint events and the random variables  $\chi_{\text{AW}(\theta),r}^i$  for  $r \in \{1, 2\}$  do not affect the expected meeting time when conditioning on  $\mathbf{R}_{\text{AW}(\theta)}^3$ . Since  $\mathbb{P}[\chi_{\text{AW}(\theta),3}^1 \neq \chi_{\text{AW}(\theta),3}^2 \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] = 0$ , this yields

$$\begin{aligned} & \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \neg(\chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3)] - \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] \\ & = (\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta),3}^1 \neq \chi_{\text{AW}(\theta),3}^2] - \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta),3}^1 = \chi_{\text{AW}(\theta),3}^2 = \mathbf{M}]) \\ & \quad \cdot \mathbb{P}[\chi_{\text{AW}(\theta),3}^1 \neq \chi_{\text{AW}(\theta),3}^2 \mid \mathbf{R}_{\text{AW}(\theta)}^3, \neg(\chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3)]. \end{aligned}$$

The last expression is non-negative because

$$\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta),3}^1 \neq \chi_{\text{AW}(\theta),3}^2] - 2(n-1) = \frac{n}{2}, \quad (9)$$

$$\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3, \chi_{\text{AW}(\theta),3}^1 = \chi_{\text{AW}(\theta),3}^2 = \mathbf{M}] - 2(n-1) = \varphi_0(n-1) = n \left( 1 - \frac{\hat{d}_n^0}{1 - \hat{d}_{n-1}^0} \right) \quad (10)$$

by Lemma 7, and

$$\begin{aligned} n \left( \frac{1}{2} - 1 + \frac{\hat{d}_n^0}{1 - \hat{d}_{n-1}^0} \right) &= \frac{n}{2(1 - \hat{d}_{n-1}^0)} (2\hat{d}_n^0 + \hat{d}_{n-1}^0 - 1) \\ &\geq \frac{n-1}{2(1 - \hat{d}_{n-1}^0)} \left( \frac{3}{e} - \frac{1}{2n!} - \frac{1}{2(n-1)!} - 1 \right) > 0 \end{aligned}$$

by Lemma 1 and because the last expression in parentheses is equal to  $\frac{3}{e} - \frac{41}{40} > 0$  for  $n = 5$  and trivially increasing in  $n$ . This shows (8), which we will now use to complete the proof.

We claim that

$$\begin{aligned} &\mathbb{E}[t_{\text{AW}(\theta)}] - \mathbb{E}[t_{\text{FSD}(\theta)}] \\ &= (\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3] - \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^{\leq 2}]) \mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^3 \mid \mathbf{R}_{\text{AW}(\theta)}^{\leq 3}] \\ &\quad - (\mathbb{E}[t_{\text{FSD}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3] - \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^{\leq 2}]) \mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^3 \mid \mathbf{R}_{\text{FSD}(\theta)}^{\leq 3}] \\ &\quad + 3(n-1) \left( \frac{1}{\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3}]} - \frac{1}{\mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3}]} \right) \\ &\geq (\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3] - \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^{\leq 2}]) (\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^3 \mid \mathbf{R}_{\text{AW}(\theta)}^{\leq 3}] - \mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^3 \mid \mathbf{R}_{\text{FSD}(\theta)}^{\leq 3}]) \\ &\quad + 3(n-1) \left( \frac{1}{\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3}]} - \frac{1}{\mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3}]} \right) \\ &= (\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3] - \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^{\leq 2}]) \left( \frac{\mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 2}]}{\mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3}]} - \frac{\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 2}]}{\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3}]} \right) \\ &\quad + 3(n-1) \left( \frac{1}{\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3}]} - \frac{1}{\mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3}]} \right) \\ &= \left( 3(n-1) - (\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3] - \mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^{\leq 2}]) \mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 2}] \right) \\ &\quad \left( \frac{1}{\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3}]} - \frac{1}{\mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3}]} \right). \end{aligned}$$

Indeed, the first equality holds by (5) and the fact that  $\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^{\leq 2}] = \mathbb{E}[t_{\text{FSD}(\theta)} \mid \mathbf{R}_{\text{FSD}(\theta)}^{\leq 2}]$  by definition of the strategies, the inequality by (8), and the last two equalities by rearranging and using that  $\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 2}] = \mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 2}]$ . Since  $\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^3] \leq 2(n-1) + n/2$  by (9) and (10),  $\mathbb{E}[t_{\text{AW}(\theta)} \mid \mathbf{R}_{\text{AW}(\theta)}^{\leq 2}] \geq 1$ , and  $\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 2}] \leq 1$ , we conclude that

$$\mathbb{E}[t_{\text{AW}(\theta)}] - \mathbb{E}[t_{\text{FSD}(\theta)}] \geq \frac{n-1}{2} \left( \frac{1}{\mathbb{P}[\mathbf{R}_{\text{AW}(\theta)}^{\leq 3}]} - \frac{1}{\mathbb{P}[\mathbf{R}_{\text{FSD}(\theta)}^{\leq 3}]} \right) \geq \frac{n-1}{2} \cdot \frac{967(1-\theta)^6}{(n-1)^9} > \frac{483(1-\theta)^6}{(n-1)^8},$$

where the second inequality follows from Theorem 1 and the fact that  $x - y > z$  for values  $x, y, z \in (0, 1)$  implies that  $\frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy} > z$ . This completes the proof.  $\square$

## A Proofs Deferred from Section 2

### A.1 Proof of Lemma 1

**Lemma 1.** *For all  $n \in \mathbb{N}_0$  and  $k \in \{0, \dots, n\}$ , the entry  $d_n^k$  is equal to the number of permutations  $\pi \in \mathcal{P}(\{1+k, \dots, n+k\})$  with  $\pi(i) \neq i$  for all  $i \in [n]$ .*

*Proof.* We write the entries in the table as  $d_n^{n-\ell}$  for  $n \in \mathbb{N}_0$  and  $\ell \in \{0, \dots, n\}$  and prove the statement by induction over  $\ell$ . For  $\ell = 0$ , the statement clearly holds since all permutations  $\pi \in \mathcal{P}(\{1+n, \dots, 2n\})$  do not have a fixed point and there are  $d_n^{n-0} = n!$  of them. Suppose that the result is correct for all entries up to a fixed value of  $\ell$ . Let  $n \in \mathbb{N}_0$  with  $n \geq \ell + 1$  be arbitrary and consider the entry  $d_n^{n-\ell-1} = d_n^{n-\ell} - d_{n-1}^{n-\ell-1}$ . For a set  $S$ , we let  $\mathcal{D}(S) := \{\pi \in \mathcal{P}(S) : \pi(i) \neq i \text{ for all } i \in [n]\}$  denote the set of derangements on  $S$ . Note that, by the induction hypothesis,  $|\mathcal{D}(\{n-\ell+1, \dots, 2n-\ell\})| = d_n^{n-\ell}$ , and that this set can be partitioned into the sets

$$\begin{aligned} \mathcal{D}' &:= \{\pi \in \mathcal{D}(\{n-\ell+1, \dots, 2n-\ell\}) : \pi(n-\ell) = 2n-\ell\}, \text{ and} \\ \mathcal{D}'' &:= \{\pi \in \mathcal{D}(\{n-\ell+1, \dots, 2n-\ell\}) : \pi(n-\ell) \neq 2n-\ell\}. \end{aligned}$$

By removing the element  $2n-\ell$ , there is a bijection between  $\mathcal{D}'$  and  $\mathcal{D}(\{n-\ell, \dots, 2n-\ell-2\})$ , whose size is  $d_{n-1}^{n-\ell-1}$  by the induction hypothesis. On the other hand, by identifying the element  $2n-\ell$  with the element  $n-\ell$  there is also a bijection between  $\mathcal{D}''$  and  $\mathcal{D}(\{n-\ell, \dots, 2n-\ell-1\})$ . We conclude that the size of this set is  $d_n^{n-\ell} - d_{n-1}^{n-\ell-1} = d_n^{n-\ell-1}$ . This shows the result.  $\square$

### A.2 Proof of Lemma 2

**Lemma 2.** *Let  $n \in \mathbb{N}$ . Then*

1.  $d_n^0 = \lfloor \frac{n!}{e} + \epsilon \rfloor$  for any  $\epsilon \in [\frac{1}{3}, \frac{1}{2}]$ ;
2.  $d_n^1 = \lfloor \frac{n!+(n-1)!}{e} + \epsilon \rfloor$  for any  $\epsilon \in [\frac{1}{3}, \frac{7}{8}]$ ;
3.  $d_n^2 = \lfloor \frac{n!+2(n-1)!+(n-2)!}{e} + \epsilon \rfloor$  for any  $\epsilon \in [\frac{1}{3}, \frac{7}{8}]$ .

*Proof.* For  $n \in \mathbb{N}$ , we have

$$d_n^0 = n! \sum_{j=0}^n \frac{(-1)^j}{j!} = n! \left( \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} - \sum_{j=n+1}^{\infty} \frac{(-1)^j}{j!} \right) = \frac{n!}{e} + \sum_{j=n+1}^{\infty} (-1)^{j+1} \frac{n!}{j!}.$$

If  $n$  is even, this yields  $d_n^0 > \frac{n!}{e}$  and  $d_n^0 = \lfloor \frac{n!}{e} + r \rfloor$  for  $r \in [\frac{1}{n+1}, 1]$ . If  $n$  is odd, we have  $d_n^0 < \frac{n!}{e}$  and  $d_n^0 = \lfloor \frac{n!}{e} + r \rfloor$  whenever  $r \leq 1 - \frac{1}{n+1}$ . Both inequalities are satisfied for  $r \in [\frac{1}{3}, \frac{1}{2}]$ .

For  $n \in \mathbb{N}$ , we further have

$$\begin{aligned} d_n^1 &= d_n^0 + d_{n-1}^0 \\ &= n! \sum_{j=0}^n \frac{(-1)^j}{j!} + (n-1)! \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} \\ &= \frac{n!}{e} + \sum_{j=n+1}^{\infty} (-1)^{j+1} \frac{n!}{j!} + \frac{(n-1)!}{e} + \sum_{j=n}^{\infty} (-1)^{j+1} \frac{(n-1)!}{j!} \\ &= \frac{n! + (n-1)!}{e} + (-1)^{n+1} \frac{1}{n} + \sum_{j=n+1}^{\infty} (-1)^{j+1} \frac{n! + (n-1)!}{j!}. \end{aligned}$$

Using that  $\frac{n!+(n-1)!}{(n+1)!} = \frac{1}{n}$  this simplifies to

$$d_n^1 = \frac{n! + (n-1)!}{e} + \sum_{j=n+2}^{\infty} (-1)^{j+1} \frac{n! + (n-1)!}{j!}.$$

If  $n$  is odd, this yields  $d_n^1 > \frac{n!+(n-1)!}{e}$  and  $d_n^1 = \lfloor \frac{n!+(n-1)!}{e} + r \rfloor$  for  $r \in [\frac{1}{(n+1)(n+2)} + \frac{1}{n(n+1)(n+2)}, 1]$ .  
If  $n$  is even, we have  $d_n^1 < \frac{n!+(n-1)!}{e}$  and  $d_n^1 = \lfloor \frac{n!+(n-1)!}{e} + r \rfloor$  for  $r \leq 1 - \frac{1}{(n+1)(n+2)} - \frac{1}{n(n+1)(n+2)}$ .  
Both inequalities are satisfied for  $r \in [\frac{1}{3}, \frac{7}{8}]$ .

For  $n \in \mathbb{N}$  with  $n \geq 2$ , we further have

$$\begin{aligned} d_n^2 &= d_n^1 + d_{n-1}^1 \\ &= d_n^0 + 2d_{n-1}^0 + d_{n-2}^0 \\ &= n! \sum_{j=0}^n \frac{(-1)^j}{j!} + 2(n-1)! \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} + (n-2)! \sum_{j=0}^{n-2} \frac{(-1)^j}{j!} \\ &= \frac{n! + 2(n-1)! + (n-2)!}{e} + \sum_{j=n+1}^{\infty} (-1)^{j+1} \frac{n!}{j!} + 2 \sum_{j=n}^{\infty} (-1)^{j+1} \frac{(n-1)!}{j!} + \sum_{j=n-1}^{\infty} (-1)^{j+1} \frac{(n-2)!}{j!} \\ &= \frac{n! + 2(n-1)! + (n-2)!}{e} + (-1)^n \frac{1}{n-1} + (-1)^{n+1} \left( \frac{2}{n} + \frac{1}{n(n+1)} \right) \\ &\quad + \sum_{j=n+1}^{\infty} (-1)^{j+1} \frac{n! + 2(n-1)! + (n-2)!}{j!}. \end{aligned}$$

It is straightforward to check that

$$\frac{1}{n+1} + \frac{2}{n(n+1)} + \frac{1}{(n-1)n(n+1)} + \frac{1}{n-1} - \left( \frac{2}{n} + \frac{1}{n(n+1)} \right) = \frac{1}{(n+1)(n-1)}.$$

Using this equality, we obtain

$$d_n^2 = \frac{n! + 2(n-1)! + (n-2)!}{e} + (-1)^n \frac{1}{(n+1)(n-1)} + \sum_{j=n+2}^{\infty} (-1)^{j+1} \frac{n! + 2(n-1)! + (n-2)!}{j!}.$$

It can be checked that for  $n \geq 2$  we have that

$$\frac{1}{(n+1)(n-1)} > \frac{1}{(n+1)(n+2)} + \frac{2}{n(n+1)(n+2)} + \frac{1}{(n-1)n(n+1)(n+2)}.$$

As a consequence, we obtain that if  $n$  is even, we have  $d_n^2 > \frac{n!+2(n-1)!+(n-2)!}{e}$  and

$$d_n^2 = \left\lfloor \frac{n! + 2(n-1)! + (n-2)!}{e} + r \right\rfloor$$

for  $r \in [\frac{1}{(n+1)(n-1)}, 1]$ . If  $n$  is odd, we have  $d_n^2 < \frac{n!+2(n-1)!+(n-2)!}{e}$  and the formula is satisfied whenever  $r \leq 1 - \frac{1}{(n+1)(n-1)}$ . Since  $n \geq 2$ , both formulas are satisfied for  $r \in [\frac{1}{3}, \frac{7}{8}]$ .  $\square$

## B Proofs Deferred from Section 3

### B.1 Proof of Proposition 1

**Proposition 1.** *Let  $G = (V, E)$  be a vertex-transitive  $(m, q)$ -clique-partitioned graph, where  $m \geq 3$  and  $q \geq 3$ . Then,  $\mathbb{P}[t_{\text{CLI}_1(\mu)} \leq 1] \leq \mathbb{P}[t_{\text{UNI}} \leq 1]$  for every  $\mu \in [0, 1]$ , and  $\mathbb{P}[t_{\text{CLI}_2(1/m)} \leq r] > \mathbb{P}[t_{\text{UNI}} \leq r]$  for every  $r \in \mathbb{N}$  with  $r \geq 2$ .*

*Proof.* Let  $G$  be a graph as in the statement. For any  $r \in \mathbb{N}$ , the probability of not meeting after  $r$  steps with the uniform strategy is  $\delta^r$ , where we recall that  $\delta$  denotes the degree of all vertices in the complement graph of  $G$  divided by  $|V|$ . For SFSD, we need to take two probabilities into account. On the one hand, the probability that two vertices chosen uniformly at random from the same clique do not have an edge between them is trivially 0. On the other hand, since vertices are fully connected to other vertices of the same clique, the probability that two vertices chosen uniformly at random from different cliques do not have an edge between them is the solution  $x$  to the equation  $\frac{1}{m} \cdot 0 + \frac{m-1}{m} \cdot x = \delta$ , which is  $x = \frac{m}{m-1}\delta$ .

To show the first inequality in the statement, we observe that

$$\mathbb{P}[t_{\text{CLI}_1(\mu)} > 1] = \frac{m}{m-1}\delta \left[ \mu^2 + 2\mu(1-\mu)\frac{m-2}{m-1} + (1-\mu)^2 \left( \frac{1}{m-1} + \left( \frac{m-2}{m-1} \right)^2 \right) \right], \quad (11)$$

because the players meet for sure if they visit the same clique in step 1, and do not meet with probability  $\frac{m}{m-1}\delta$  if they visit different cliques in step 1, which occurs

- (i) with probability 1 if both stay in the clique where they started;
- (ii) with probability  $1 - 1/(m-1)$  if exactly one of them moves to another clique, because this is the other player's clique with probability  $1/(m-1)$ ; and
- (iii) with probability  $\frac{1}{m-1} + \left(\frac{m-2}{m-1}\right)^2$  if both move to another clique, because they do not meet with probability 1 when one player visits the other player's former clique and with probability  $(m-2)/(m-1)$  otherwise.

The expression on the right-hand side of (11) is not larger than  $\mathbb{P}[t_{\text{UNI}} > 1] = \delta$  if and only if

$$\begin{aligned} \frac{m}{m-1} \left[ \mu^2 + 2\mu(1-\mu)\frac{m-2}{m-1} + (1-\mu)^2 \left( \frac{1}{m-1} + \left( \frac{m-2}{m-1} \right)^2 \right) \right] &\leq 1 \\ \iff m(m-1)^2\mu^2 + 2m(m-1)(m-2)\mu(1-\mu) + m(m^2 - 3m + 3)(1-\mu)^2 &\leq (m-1)^3 \\ \iff m((m-1)^2 - 2(m-1)(m-2) + (m^2 - 3m + 3))\mu^2 & \\ + 2m((m-1)(m-2) - (m^2 - 3m + 3))\mu + m(m^2 - 3m + 3) - (m-1)^3 &\leq 0 \\ \iff (m\mu - 1)^2 &\leq 0, \end{aligned}$$

which holds with equality if and only if  $\mu = 1/m$ . For any other value of  $\mu$ , we have the expression on the right-hand side of (11) is strictly larger than  $\delta$ . This finishes the proof of the first inequality in the statement.

For the second inequality, it suffices to prove that  $\mathbb{P}[t_{\text{CLI}_2(1/m)} > 2] < \delta^2$ . Indeed, if this holds, then for every  $k \in \mathbb{N}$  and  $r \in \{0, 1\}$  we have

$$\mathbb{P}[t_{\text{CLI}_2(1/m)} > 2k + r] = \mathbb{P}[t_{\text{CLI}_2(1/m)} > 2]^k \delta^r < \delta^{2k+r} = \mathbb{P}[t_{\text{UNI}} > 2k + r],$$

where the first equality holds because  $\text{CLI}_2(1/m)$  restarts and repeats after every pair of steps. Thus, we focus on proving that  $\mathbb{P}[t_{\text{CLI}_2(1/m)} > 2] < \delta^2$  holds in what follows.

We first compute the probability that the players do not coincide in the same clique after two steps under  $\text{CLI}_2(\mu)$ , conditional on both players moving. By conditioning on the number of players visiting the home clique of the other in the first step, we obtain that this probability equals

$$\begin{aligned} & \left(\frac{1}{m-1}\right)^2 \frac{m-3}{m-2} + \frac{2(m-2)}{(m-1)^2} \left(\frac{1}{m-2} + \left(\frac{m-3}{m-2}\right)^2\right) \\ & + \frac{(m-2)(m-3)}{(m-1)^2} \left(\frac{2}{m-2} + \frac{(m-4)(m-3)}{(m-2)^2}\right) = \frac{m^3 - 6m^2 + 14m - 13}{(m-1)^2(m-2)}. \end{aligned}$$

Indeed, the first term corresponds to both players visiting the other's home clique in the first step. This event occurs with probability  $1/(m-1)^2$  and, conditional on it, both players choose a clique uniformly at random in the second step from the same set of  $m-2$  cliques, so they choose different ones with probability  $(m-3)/(m-2)$ . The second term corresponds to exactly one player visiting the other's home clique in the first step, which occurs with probability  $2(m-2)/(m-1)^2$ . Conditional on this event, say w.l.o.g. that player 1 visits the home clique of player 2 in the first step, the players do not meet in the second step if either player 2 visits the home clique of player 1, which occurs with probability  $1/(m-2)$ , or both visit different cliques that are not the home clique of any of them, which occurs with probability  $(m-3)^2/(m-2)^2$ . The third term corresponds to the players visiting different cliques that are not the home clique of any of them in the first step, which occurs with probability  $(m-2)(m-3)/(m-1)^2$ . Conditional on this event, the players do not meet in the second step if either player 1 visits a clique that player 2 already visited, which occurs with probability  $2/(m-2)$ , or player 1 visits another clique but player 2 does not choose the same one, which occurs with probability  $(m-3)(m-4)/(m-2)^2$ .

We can now compute the non-meeting probability  $\text{CLI}_2(\mu)$  after two steps similarly to the case before, by conditioning on the number of players that move across cliques:

$$\mathbb{P}[t_{\text{CLI}_2(\mu)} > 2] = \left(\frac{m}{m-1}\right)^2 \delta^2 \left[ \mu^2 + 2\mu(1-\mu)\frac{m-3}{m-1} + (1-\mu)^2 \frac{m^3 - 6m^2 + 14m - 13}{(m-1)^2(m-2)} \right].$$

This expression is strictly smaller than  $\mathbb{P}[t_{\text{UNI}} > 2] = \delta^2$  if and only if

$$\begin{aligned} & \left(\frac{m}{m-1}\right)^2 \left[ \mu^2 + 2\mu(1-\mu)\frac{m-3}{m-1} + (1-\mu)^2 \frac{m^3 - 6m^2 + 14m - 13}{(m-1)^2(m-2)} \right] < 1 \\ & \iff m^2(m-1)^2(m-2)\mu^2 + 2m^2(m-1)(m-2)(m-3)\mu(1-\mu) \\ & \quad + m^2(m^3 - 6m^2 + 14m - 13)(1-\mu)^2 < (m-1)^4(m-2) \\ & \iff m^2((m-1)^2(m-2) - 2(m-1)(m-2)(m-3) + (m^3 - 6m^2 + 14m - 13))\mu^2 \\ & \quad + 2m^2((m-1)(m-2)(m-3) - (m^3 - 6m^2 + 14m - 13))\mu \\ & \quad + m^2(m^3 - 6m^2 + 14m - 13) - (m-1)^4(m-2) < 0 \\ & \iff m^2(2m^2 - 3m - 3)\mu^2 - 2m^2(3m - 7)\mu + 3m^2 - 9m + 2 < 0. \end{aligned}$$

Taking  $\mu = 1/m$ , the left-hand side becomes

$$2m^2 - 3m - 3 - 6m^2 + 14m + 3m^2 - 9m + 2 = -(m-1)^2,$$

which is strictly negative for any  $m > 1$ . □

## C Proofs Deferred from Section 4.1

### C.1 Proof of Lemma 3

**Lemma 3.** For  $n \in \mathbb{N}$  with  $n \geq 4$ , and  $i \in \{1, 2\}$ ,

$$w_1^i = \frac{1}{(n-1)^2}, \quad w_2^i = \frac{n-2}{(n-1)^2}, \quad w_3^i = \frac{3(n-2)}{(n-1)^2}, \quad w_4^i = \frac{(n-2)(n-4)}{(n-1)^2}.$$

*Proof.* Fix  $n$  and  $i$  as in the statement, and  $\theta \in [0, 1)$  arbitrarily. Conditional on moving in the first three rounds, the locations  $\pi_{\text{FSD}(\theta),1}^i(1)$  and  $\pi_{\text{FSD}(\theta),2}^i(1)$  are chosen uniformly at random among the non-home locations of player  $i$ , i.e., among  $[n] \setminus \{i\}$ . In addition,  $\pi_{\text{FSD}(\theta),3}^i(1)$  is set to the same location  $\pi_{\text{FSD}(\theta),1}^i$  if  $\pi_{\text{FSD}(\theta),1}^i(1) = \pi_{\text{FSD}(\theta),2}^i(1)$  and to a different location in  $[n] \setminus \{i, \pi_{\text{FSD}(\theta),1}^i(1), \pi_{\text{FSD}(\theta),2}^i(1)\}$  chosen uniformly at random, otherwise. Thus, we obtain

$$\begin{aligned} w_1^i &= \mathbb{P}[\pi_{\text{FSD}(\theta),1}^i(1) = \pi_{\text{FSD}(\theta),2}^i(1) = 3-i] \\ &= \mathbb{P}[\pi_{\text{FSD}(\theta),1}^i(1) = 3-i] \mathbb{P}[\pi_{\text{FSD}(\theta),2}^i(1) = 3-i] = \frac{1}{(n-1)^2}, \\ w_2^i &= \mathbb{P}[\pi_{\text{FSD}(\theta),1}^i(1) = \pi_{\text{FSD}(\theta),2}^i(1) \neq 3-i] \\ &= \mathbb{P}[\pi_{\text{FSD}(\theta),1}^i(1) \neq 3-i] \mathbb{P}[\pi_{\text{FSD}(\theta),2}^i(1) = \pi_{\text{FSD}(\theta),1}^i(1)] = \frac{n-2}{n-1} \cdot \frac{1}{n-1} = \frac{n-2}{(n-1)^2}, \\ w_3^i &= \mathbb{P}[\pi_{\text{FSD}(\theta),1}^i(1) \neq \pi_{\text{FSD}(\theta),2}^i(1)] \\ &\quad \cdot \mathbb{P}[3-i \in \{\pi_{\text{FSD}(\theta),1}^i(1), \pi_{\text{FSD}(\theta),2}^i(1), \pi_{\text{FSD}(\theta),3}^i(1)\} \mid \pi_{\text{FSD}(\theta),1}^i(1) \neq \pi_{\text{FSD}(\theta),2}^i(1)] \\ &= \frac{n-2}{n-1} \cdot \frac{3}{n-1} = \frac{3(n-2)}{(n-1)^2}, \\ w_4^i &= \mathbb{P}[\pi_{\text{FSD}(\theta),1}^i(1) \neq \pi_{\text{FSD}(\theta),2}^i(1)] \\ &\quad \cdot \mathbb{P}[3-i \notin \{\pi_{\text{FSD}(\theta),1}^i(1), \pi_{\text{FSD}(\theta),2}^i(1), \pi_{\text{FSD}(\theta),3}^i(1)\} \mid \pi_{\text{FSD}(\theta),1}^i(1) \neq \pi_{\text{FSD}(\theta),2}^i(1)] \\ &= \frac{n-2}{n-1} \cdot \frac{n-4}{n-1} = \frac{(n-2)(n-4)}{(n-1)^2}. \quad \square \end{aligned}$$

### C.2 Proof of Lemma 4

**Lemma 4.** For every  $n \in \mathbb{N}$  with  $n \geq 4$ , and  $\theta \in [0, 1)$ ,

$$\begin{aligned} \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] &= \\ &= \frac{1}{(n-1)^4(n-3)} \left( (n-3)(\hat{d}_{n-2}^0)^3 + 6(n-2)(n-3)\hat{d}_{n-2}^0(\hat{d}_{n-2}^1)^2 \right. \\ &\quad + 3(n-2)(n^2 - 7n + 13)\hat{d}_{n-2}^0(\hat{d}_{n-2}^2)^2 \\ &\quad + 2(n-2)(n-3)^2(\hat{d}_{n-2}^1)^3 + 6(n-2)(n-3)^2(\hat{d}_{n-2}^1)^2\hat{d}_{n-2}^2 \\ &\quad + 6(n-2)(n-4)(n^2 - 6n + 10)\hat{d}_{n-2}^1(\hat{d}_{n-2}^2)^2 \\ &\quad \left. + (n^5 - 16n^4 + 103n^3 - 335n^2 + 551n - 362)(\hat{d}_{n-2}^2)^3 \right). \end{aligned}$$

In particular, this value equals  $\frac{1}{8}$  for  $n = 4$ ,  $\frac{5}{54}$  for  $n = 5$ ,  $\frac{439471}{5184000}$  for  $n = 6$ , and  $\frac{406417}{5184000}$  for  $n = 7$ .

*Proof.* We fix  $n \in \mathbb{N}$  with  $n \geq 4$ , and we begin by applying total probabilities to obtain

$$\mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^3 \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] = \sum_{k, \ell \in [4]} \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^3 \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3, A_k^1, A_\ell^2] \cdot w_k^0 \cdot w_\ell^1. \quad (12)$$

We denote the first term in the sum by

$$Z_{k\ell} := \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^3 \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3, A_k^1, A_\ell^2],$$

corresponding to the non-meeting probability under  $\text{FSD}(\theta)$  conditional on both players moving in the first three rounds and on the events  $A_k^1$  and  $A_\ell^2$ , for  $k, \ell \in [4]$ . It is clear that  $Z \in [0, 1]^{4 \times 4}$  is a symmetric matrix; we now compute its entries on and above the diagonal. We repeatedly use Lemma 1 to compute the non-meeting probabilities in steps  $2, \dots, n-1$  of each round conditional on the first locations: if these first locations are each other's home locations, the corresponding non-meeting probability is  $\hat{d}_{n-2}^0$ , if exactly one of these first locations is the other player's home location, the corresponding non-meeting probability is  $\hat{d}_{n-2}^1$ ; if none of these first locations are each other's home locations and they are different, the corresponding non-meeting probability is  $\hat{d}_{n-2}^2$ .

When we condition on  $A_1^1$ , we have  $\pi_{\text{FSD}(\theta), r}^1(1) = 2$  and thus  $\pi_{\text{FSD}(\theta), r}^1(1) \neq \pi_{\text{FSD}(\theta), r}^2(1)$  for each  $r \in [3]$ . If  $\pi_{\text{FSD}(\theta), r}^2(1) = 1$  for a round  $r \in [3]$ , the meeting probability in later steps of that round (between  $(r-1)(n-1)+2$  and  $r(n-1)$ ) is  $1 - \hat{d}_{n-2}^0$ . Similarly, if  $\pi_{\text{FSD}(\theta), r}^2(1) \neq 1$  for a round  $r \in [3]$ , the meeting probability in later steps of that round is  $1 - \hat{d}_{n-2}^1$ . Therefore,

$$Z_{11} = (\hat{d}_{n-2}^0)^3, \quad Z_{12} = (\hat{d}_{n-2}^1)^3, \quad Z_{13} = \hat{d}_{n-2}^0 (\hat{d}_{n-2}^1)^2, \quad Z_{14} = (\hat{d}_{n-2}^1)^3.$$

When we condition on  $A_2^1$ , we have  $\pi_{\text{FSD}(\theta), r}^1(1) \neq 2$  for all  $r \in [3]$ , so the event  $\pi_{\text{FSD}(\theta), r}^1(1) = \pi_{\text{FSD}(\theta), r}^2(1)$  for some  $r \in [3]$  has non-zero probability. Indeed, it occurs with probability  $1/(n-2)$  conditional on  $A_2^2$ , with probability  $2/(n-2)$  conditional on  $A_3^2$ , and with probability  $3/(n-2)$  conditional on  $A_4^2$ . In addition, if  $\pi_{\text{FSD}(\theta), r}^2(1) = 1$  for a round  $r \in [3]$ , the meeting probability in later steps of that round is  $1 - \hat{d}_{n-2}^1$ . If  $\pi_{\text{FSD}(\theta), r}^2(1) \neq 1$  for a round  $r \in [3]$ , the meeting probability in later steps of that round, conditional on  $\pi_{\text{FSD}(\theta), r}^1(1) \neq \pi_{\text{FSD}(\theta), r}^2(1)$ , is  $1 - \hat{d}_{n-2}^2$ . Therefore,

$$Z_{22} = \frac{n-3}{n-2} (\hat{d}_{n-2}^2)^3, \quad Z_{23} = \frac{n-4}{n-2} \hat{d}_{n-2}^1 (\hat{d}_{n-2}^2)^2, \quad Z_{24} = \frac{n-5}{n-2} (\hat{d}_{n-2}^2)^3.$$

When we condition on  $A_3^1$  and  $A_3^2$ , we have  $\pi_{\text{FSD}(\theta), r}^1(1) = 2$  and  $\pi_{\text{FSD}(\theta), s}^2(1) = 1$  for some rounds  $r, s \in [3]$ , so we need to further condition on whether  $r = s$ , which occurs with probability  $1/3$ , or  $r \neq s$ , which occurs with probability  $2/3$ . If  $r = s$ , we have  $\pi_{\text{FSD}(\theta), r}^1(1) \neq \pi_{\text{FSD}(\theta), r}^2(1)$  and the meeting probability in later steps of that round is  $1 - \hat{d}_{n-2}^0$ . The event  $\pi_{\text{FSD}(\theta), r'}^1(1) = \pi_{\text{FSD}(\theta), r'}^2(1)$  for some other round  $r' \in [3] \setminus \{r\}$  occurs with probability  $\frac{1}{n-2} + \frac{n-4}{n-2} \cdot \frac{1}{n-3}$ . The meeting probability in later steps of a round  $r' \in [3] \setminus \{r\}$ , conditional on  $\pi_{\text{FSD}(\theta), r'}^1(1) \neq \pi_{\text{FSD}(\theta), r'}^2(1)$ , is  $1 - \hat{d}_{n-2}^2$ . If  $r \neq s$ , we have  $\pi_{\text{FSD}(\theta), r'}^1(1) \neq \pi_{\text{FSD}(\theta), r'}^2(1)$  for  $r' \in \{r, s\}$ , and the meeting probability in later steps of these rounds is  $1 - \hat{d}_{n-2}^1$ . The event  $\pi_{\text{FSD}(\theta), r'}^1(1) = \pi_{\text{FSD}(\theta), r'}^2(1)$  for the round  $r' \in [3] \setminus \{r, s\}$  occurs with probability  $\frac{1}{n-2}$ . The meeting probability in later steps of the

round  $r' \in [3] \setminus \{r, s\}$ , conditional on  $\pi_{\text{FSD}(\theta),r'}^1(1) \neq \pi_{\text{FSD}(\theta),r'}^2(1)$ , is  $1 - \hat{d}_{n-2}^2$ . Combining these probabilities, we obtain

$$Z_{33} = \frac{1}{3} \left( 1 - \left( \frac{1}{n-2} + \frac{n-4}{n-2} \cdot \frac{1}{n-3} \right) \right) \hat{d}_{n-2}^0 (\hat{d}_{n-2}^2)^2 + \frac{2}{3} \left( 1 - \frac{1}{n-2} \right) (\hat{d}_{n-2}^1)^2 \hat{d}_{n-2}^2.$$

When we condition on  $A_3^1$  and  $A_4^2$ , we have  $\pi_{\text{FSD}(\theta),r}^1(1) = 2$  for some round  $r \in [3]$ . This implies  $\pi_{\text{FSD}(\theta),r}^1(1) \neq \pi_{\text{FSD}(\theta),r}^2(1)$  and the meeting probability in later steps of that round is  $1 - \hat{d}_{n-2}^1$ . The event  $\pi_{\text{FSD}(\theta),s}^1(1) = \pi_{\text{FSD}(\theta),s}^2(1)$  for some other round  $s \in [3] \setminus \{r\}$  occurs with probability  $\frac{1}{n-2} + \frac{n-4}{n-2} \cdot \frac{1}{n-3}$ . The meeting probability in later steps of a round  $s \in [3] \setminus \{r\}$ , conditional on  $\pi_{\text{FSD}(\theta),s}^1(1) \neq \pi_{\text{FSD}(\theta),s}^2(1)$ , is  $1 - \hat{d}_{n-2}^2$ . Therefore,

$$Z_{34} = \left( 1 - \left( \frac{1}{n-2} + \frac{n-4}{n-2} \cdot \frac{1}{n-3} \right) \right) \hat{d}_{n-2}^1 (\hat{d}_{n-2}^2)^2.$$

Finally, when we condition on  $A_4^1$  and  $A_4^2$ , we compute the probability that  $\pi_{\text{FSD}(\theta),r}^1(1) \neq \pi_{\text{FSD}(\theta),r}^2(1)$  for every round  $r \in [3]$  by distinguishing whether the event  $\pi_{\text{FSD}(\theta),1}^2(1) \in \{\pi_{\text{FSD}(\theta),2}^1(1), \pi_{\text{FSD}(\theta),3}^1(1)\}$  holds, which occurs with probability  $2/(n-2)$ . If it does hold, say w.l.o.g.  $\pi_{\text{FSD}(\theta),1}^2(1) = \pi_{\text{FSD}(\theta),2}^1(1)$ , then  $\pi_{\text{FSD}(\theta),r}^1(1) \neq \pi_{\text{FSD}(\theta),r}^2(1)$  for  $r \in [2]$ , and we have  $\pi_{\text{FSD}(\theta),r}^1(1) \neq \pi_{\text{FSD}(\theta),r}^2(1)$  for every round  $r \in [3]$  whenever  $\pi_{\text{FSD}(\theta),3}^1(1) \neq \pi_{\text{FSD}(\theta),3}^2(1)$ , which occurs with probability  $(n-4)/(n-3)$  conditional on the former event. Conversely, conditional on the fact that  $\pi_1^2(1) \notin \{\pi_2^1(1), \pi_3^1(1)\}$ , we have  $\pi_{\text{FSD}(\theta),r}^1(1) \neq \pi_{\text{FSD}(\theta),r}^2(1)$  for every round  $r \in [3]$  if either  $\pi_{\text{FSD}(\theta),2}^2(1) = \pi_{\text{FSD}(\theta),3}^1(1)$  holds, which occurs with probability  $1/(n-3)$ , or if both  $\pi_{\text{FSD}(\theta),2}^2(1) \notin \{\pi_{\text{FSD}(\theta),2}^1(1), \pi_{\text{FSD}(\theta),3}^1(1)\}$  and  $\pi_{\text{FSD}(\theta),3}^2(1) \neq \pi_{\text{FSD}(\theta),3}^1(1)$  hold, which occurs with overall probability  $\frac{n-5}{n-3} \cdot \frac{n-5}{n-4}$ . Regarding the later steps of each round, conditional on having  $\pi_{\text{FSD}(\theta),r}^1(1) \neq \pi_{\text{FSD}(\theta),r}^2(1)$  for every round  $r \in [3]$ , the meeting probability in later steps of each of these rounds is  $1 - \hat{d}_{n-2}^2$ . Therefore,

$$Z_{44} = \left( \frac{n-5}{n-2} \left( \frac{n-5}{n-3} \cdot \frac{n-5}{n-4} + \frac{1}{n-3} \right) + \frac{2}{n-2} \cdot \frac{n-4}{n-3} \right) (\hat{d}_{n-2}^2)^3.$$

Combining (12) with Lemma 3 and the previous expressions for  $Z_{k\ell}$ , we obtain

$$\begin{aligned} & \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^3 \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}] \\ &= \sum_{k \in [4]} \sum_{\ell \in [4]} w_k^1 w_\ell^2 Z_{k\ell} \\ &= \frac{1}{(n-1)^4} (\hat{d}_{n-2}^0)^3 + \frac{2(n-2)}{(n-1)^4} (\hat{d}_{n-2}^1)^3 + \frac{6(n-2)}{(n-1)^4} \hat{d}_{n-2}^0 (\hat{d}_{n-2}^1)^2 + \frac{2(n-2)(n-4)}{(n-1)^4} (\hat{d}_{n-2}^1)^3 \\ &+ \frac{(n-2)^2}{(n-1)^4} \cdot \frac{n-3}{n-2} (\hat{d}_{n-2}^2)^3 + \frac{6(n-2)^2}{(n-1)^4} \cdot \frac{n-4}{n-2} \hat{d}_{n-2}^1 (\hat{d}_{n-2}^2)^2 + \frac{2(n-2)^2(n-4)}{(n-1)^4} \cdot \frac{n-5}{n-2} (\hat{d}_{n-2}^2)^3 \\ &+ \frac{9(n-2)^2}{(n-1)^4} \left[ \frac{1}{3} \left( 1 - \left( \frac{1}{n-2} + \frac{n-4}{n-2} \cdot \frac{1}{n-3} \right) \right) \hat{d}_{n-2}^0 (\hat{d}_{n-2}^2)^2 + \frac{2}{3} \left( 1 - \frac{1}{n-2} \right) (\hat{d}_{n-2}^1)^2 \hat{d}_{n-2}^2 \right] \\ &+ \frac{6(n-2)^2(n-4)}{(n-1)^4} \cdot \left( 1 - \left( \frac{1}{n-2} + \frac{n-4}{n-2} \cdot \frac{1}{n-3} \right) \right) \hat{d}_{n-2}^1 (\hat{d}_{n-2}^2)^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{(n-2)^2(n-4)^2}{(n-1)^4} \cdot \left( \frac{n-5}{n-2} \left( \frac{n-5}{n-3} \cdot \frac{n-5}{n-4} + \frac{1}{n-3} \right) + \frac{2}{n-2} \cdot \frac{n-4}{n-3} \right) (\hat{d}_{n-2}^2)^3 \\
= & \frac{1}{(n-1)^4(n-3)} \left[ (n-3)(\hat{d}_{n-2}^0)^3 + 6(n-2)(n-3)\hat{d}_{n-2}^0(\hat{d}_{n-2}^1)^2 \right. \\
& + 3(n-2)(n^2-7n+13)\hat{d}_{n-2}^0(\hat{d}_{n-2}^2)^2 + 2(n-2)(n-3)^2(\hat{d}_{n-2}^1)^3 \\
& + 6(n-2)(n-3)^2(\hat{d}_{n-2}^1)^2\hat{d}_{n-2}^2 + 6(n-2)(n-4)(n^2-6n+10)\hat{d}_{n-2}^1(\hat{d}_{n-2}^2)^2 \\
& \left. + (n^5-16n^4+103n^3-335n^2+551n-362)(\hat{d}_{n-2}^2)^3 \right].
\end{aligned}$$

For  $n \in \{4, 5, 6, 7\}$ , we evaluate this expression explicitly to obtain the values in the statement. For  $n = 4$  we have that

$$\hat{d}_3^1 = \frac{3}{6} = \frac{1}{2}, \quad \hat{d}_2^0 = \frac{1}{2}, \quad \hat{d}_2^1 = \frac{1}{2}, \quad \hat{d}_2^2 = 1,$$

so replacing in the expression above yields

$$\begin{aligned}
\mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^3 \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] = \\
\frac{1}{81} \left( \frac{1}{8} + 12 \cdot \frac{1}{2} \cdot \frac{1}{4} + 6 \cdot \frac{1}{2} \cdot 1 + 4 \cdot \frac{1}{8} + 12 \cdot \frac{1}{4} \cdot 1 + 12 \cdot 0 \cdot 2 \cdot \frac{1}{2} \cdot 1 + 2 \cdot 1 \right) = \frac{1}{8}.
\end{aligned}$$

For  $n = 5$  we have that

$$\hat{d}_4^1 = \frac{11}{24}, \quad \hat{d}_3^0 = \frac{2}{6} = \frac{1}{3}, \quad \hat{d}_3^1 = \frac{3}{6} = \frac{1}{2}, \quad \hat{d}_3^2 = \frac{4}{6} = \frac{2}{3},$$

so replacing in the expression above yields

$$\begin{aligned}
\mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^3 \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] = \\
\frac{1}{512} \left( 2 \cdot \frac{1}{27} + 36 \cdot \frac{1}{3} \cdot \frac{1}{4} + 27 \cdot \frac{1}{3} \cdot \frac{4}{9} + 24 \cdot \frac{1}{8} + 72 \cdot \frac{1}{4} \cdot \frac{2}{3} + 90 \cdot \frac{1}{2} \cdot \frac{4}{9} + 18 \cdot \frac{8}{27} \right) = \frac{5}{54}.
\end{aligned}$$

For  $n = 6$  we have that

$$\hat{d}_5^1 = \frac{53}{120}, \quad \hat{d}_4^0 = \frac{9}{24} = \frac{3}{8}, \quad \hat{d}_4^1 = \frac{11}{24}, \quad \hat{d}_4^2 = \frac{14}{24} = \frac{7}{12},$$

so replacing in the expression above yields

$$\begin{aligned}
\mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^3 \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] = \\
\frac{1}{1875} \left( 3 \cdot \frac{729}{13824} + 72 \cdot \frac{9}{24} \cdot \frac{121}{576} + 84 \cdot \frac{9}{24} \cdot \frac{196}{576} + 72 \cdot \frac{1331}{13824} \right. \\
\left. + 216 \cdot \frac{121}{576} \cdot \frac{14}{24} + 480 \cdot \frac{11}{24} \cdot \frac{196}{576} + 172 \cdot \frac{2744}{13824} \right) = \frac{439471}{5184000}.
\end{aligned}$$

For  $n = 7$  we have that

$$\hat{d}_6^1 = \frac{309}{720} = \frac{103}{240}, \quad \hat{d}_5^0 = \frac{44}{120} = \frac{11}{30}, \quad \hat{d}_5^1 = \frac{53}{120}, \quad \hat{d}_5^2 = \frac{64}{120} = \frac{8}{15},$$

so replacing in the expression above yields

$$\begin{aligned} & \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^3 \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] = \\ & \frac{1}{5184} \left( 4 \cdot \frac{85184}{1728000} + 120 \cdot \frac{44 \cdot 2809}{1728000} + 195 \cdot \frac{44 \cdot 4096}{1728000} + 160 \cdot \frac{148877}{1728000} \right. \\ & \quad \left. + 480 \cdot \frac{2809 \cdot 64}{1728000} + 1530 \cdot \frac{53 \cdot 4096}{1728000} + 800 \cdot \frac{262144}{1728000} \right) = \frac{702288576}{8957952000} = \frac{406417}{5184000}. \quad \square \end{aligned}$$

### C.3 Proof of Lemma 5

**Lemma 5.** For  $n = 4$ ,  $\mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] = (\hat{d}_{n-1}^1)^3$ . For  $n \in \mathbb{N}$  with  $n \geq 5$ ,

$$(\hat{d}_{n-1}^1)^3 - \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \geq \frac{967}{(n-1)^9}.$$

Before proving the lemma, we state a result that will be used for this purpose.

**Lemma 11.** Let  $r_1, r_2, r_3, r_4: \mathbb{N} \rightarrow \mathbb{R}$  be the functions such that, for  $m \in \mathbb{N}$ ,

$$\begin{aligned} r_1(m) &= \frac{8m^7 - 88m^6 + 360m^5 - 664m^4 + 520m^3 - 168m^2 + 184m - 152}{(m-2)e^3}, \\ r_2(m) &= \frac{512m^5 - 5120m^4 + 21504m^3 - 49125m^2 + 62868m - 36271}{1728(m-2)^2(m-4)!^3}, \\ r_3(m) &= \frac{64m^6 - 704m^5 + 3127m^4 - 7233m^3 + 9081m^2 - 5103m + 128}{24(m-2)(m-4)!^2e}, \\ r_4(m) &= \frac{8m^7 - 93m^6 + 426m^5 - 980m^4 + 1182m^3 - 631m^2 - 8m - 24}{(m-4)!e^2}. \end{aligned}$$

Then  $r_1$  is increasing and  $r_2, r_3, r_4$  are decreasing for  $m \geq 8$ .

*Proof of Lemma 11.* We compute the difference of any two consecutive terms of each function and show that it has the desired sign. For  $r_1$ , we obtain

$$r_1(m+1) - r_1(m) = \frac{48m^7 - 384m^6 + 1120m^5 - 1392m^4 + 576m^3 + 16m^2 + 176m - 152}{(m-1)(m-2)e^3}.$$

The denominator is clearly positive for every  $m \geq 8$ . As for the numerator, we can show its positivity when  $m \geq 8$  from the fact that the (positive) coefficient of each term corresponding to an odd power of  $m$ , say the  $k$ th power, times 8, is larger than the (potentially negative) coefficient corresponding to the  $(k-1)$ th power, with a strict inequality for  $k \leq 5$ . This yields, for example  $48m^7 \geq 384m^6$ ,  $1120m^5 \geq 8960m^4 > 1392m^4$ , and similarly for the smaller powers. We conclude that  $r_1$  is increasing in  $m \geq 8$ .

For  $r_2$ , we obtain

$$\begin{aligned} r_2(m) - r_2(m+1) &= \frac{(m-1)(m-2)}{1728(m-1)!^3} (512m^{10} - 10752m^9 + 101376m^8 - 567781m^7 + 2099307m^6 \\ & \quad - 5369905m^5 + 9637252m^4 - 11962957m^3 + 9774397m^2 \\ & \quad - 4702755m + 1001845). \end{aligned}$$

The term outside the parentheses is clearly positive for every  $m \geq 8$ . The term in the parentheses has the following approximate values for  $8 \leq m \leq 20$ :  $2.50286 \cdot 10^{10}$ ,  $1.21744 \cdot 10^{11}$ ,  $4.75447 \cdot 10^{11}$ ,  $1.57424 \cdot 10^{12}$ ,  $4.58175 \cdot 10^{12}$ ,  $1.20218 \cdot 10^{13}$ ,  $2.89666 \cdot 10^{13}$ ,  $6.49912 \cdot 10^{13}$ ,  $1.37247 \cdot 10^{14}$ ,  $2.75120 \cdot 10^{14}$ ,  $5.27073 \cdot 10^{14}$ ,  $9.70415 \cdot 10^{14}$ ,  $1.72494 \cdot 10^{15}$ . For  $m \geq 21$ , it is clearly positive since, similarly as before, we have that the coefficient multiplying an even power of  $m$ , say  $m^k$  with  $k \in [10]$  even, is larger than the coefficient multiplying  $m^{k-1}$  divided by 21, with strict inequality for all powers of  $m$  smaller than 8. We conclude that  $r_2$  is decreasing in  $m \geq 8$ .

For  $r_3$  we obtain

$$\begin{aligned} r_3(m) - r_3(m+1) &= \frac{1}{24(m-1)!^2 e} (64m^{11} - 1344m^{10} + 12535m^9 - 68967m^8 + 249078m^7 \\ &\quad - 618547m^6 + 1070686m^5 - 1275533m^4 + 1001527m^3 \\ &\quad - 471833m^2 + 107198m - 4864). \end{aligned}$$

The term outside the parentheses is clearly positive for every  $m \geq 8$ . The term in the parentheses has the following approximate values for  $8 \leq m \leq 20$ :  $2.25409 \cdot 10^{10}$ ,  $1.27820 \cdot 10^{11}$ ,  $5.65802 \cdot 10^{11}$ ,  $2.08630 \cdot 10^{12}$ ,  $6.67848 \cdot 10^{12}$ ,  $1.90922 \cdot 10^{13}$ ,  $4.97477 \cdot 10^{13}$ ,  $1.19962 \cdot 10^{14}$ ,  $2.70871 \cdot 10^{14}$ ,  $5.78002 \cdot 10^{14}$ ,  $1.17424 \cdot 10^{15}$ ,  $2.28484 \cdot 10^{15}$ ,  $4.27947 \cdot 10^{15}$ . For  $m \geq 21$ , it is positive for the same reason as the previous cases: each coefficient multiplying an odd power of  $m$  is larger than the coefficient multiplying the immediately smaller odd power divided by 21, with strict inequality for all powers of  $m$  smaller than 9. We conclude that  $r_3$  is decreasing in  $m \geq 8$ .

Finally, for  $r_4$  we obtain

$$r_4(m) - r_4(m+1) = \frac{8m^8 - 125m^7 + 742m^6 - 2294m^5 + 4087m^4 - 4119m^3 + 1817m^2 + 16m + 192}{(m-3)!e^2}.$$

The denominator is clearly positive for every  $m \geq 8$ . The numerator has the following approximate values for  $8 \leq m \leq 15$ :  $6.16282 \cdot 10^6$ ,  $2.9333 \cdot 10^7$ ,  $9.95331 \cdot 10^7$ ,  $2.78598 \cdot 10^8$ ,  $6.83550 \cdot 10^8$ ,  $1.52001 \cdot 10^9$ ,  $3.12883 \cdot 10^9$ ,  $6.04895 \cdot 10^9$ . For  $m \geq 16$ , it is positive for the same reason as the previous cases: each coefficient multiplying an even power of  $m$  is strictly larger than the coefficient multiplying the immediately smaller odd power divided by 16. We conclude that  $r_4$  is decreasing in  $m \geq 8$ .  $\square$

We now proceed with the proof of Lemma 5.

*Proof of Lemma 5.* By Lemma 2, for all  $m \in \mathbb{N}$ ,

$$\hat{d}_m^0 \leq \frac{1}{e} + \frac{1}{3m!}, \quad \frac{1}{e} + \frac{1}{me} - \frac{1}{8m!} \leq \hat{d}_m^1 \leq \frac{1}{e} + \frac{1}{me} + \frac{1}{3m!}, \quad \hat{d}_m^2 \leq \frac{1}{e} + \frac{2}{me} + \frac{1}{m(m-1)e} + \frac{1}{3m!}.$$

Together with Lemma 4,

$$\begin{aligned} &(\hat{d}_{n-1}^1)^3 - \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ &\geq \left( \frac{1}{e} + \frac{1}{(n-1)e} - \frac{1}{8(n-1)!} \right)^3 - \frac{1}{(n-1)^4} \left( \frac{1}{e} + \frac{1}{3(n-2)!} \right)^3 \\ &\quad - \frac{6(n-2)}{(n-1)^4} \left( \frac{1}{e} + \frac{1}{3(n-2)!} \right) \left( \frac{1}{e} + \frac{1}{(n-2)e} + \frac{1}{3(n-2)!} \right)^2 \end{aligned}$$

$$\begin{aligned}
& - \frac{3(n-2)(n^2-7n+13)}{(n-1)^4(n-3)} \left( \frac{1}{e} + \frac{1}{3(n-2)!} \right) \\
& \quad \left( \frac{1}{e} + \frac{2}{(n-2)e} + \frac{1}{(n-2)(n-3)e} + \frac{1}{3(n-2)!} \right)^2 \\
& - \frac{2(n-2)(n-3)}{(n-1)^4} \left( \frac{1}{e} + \frac{1}{(n-2)e} + \frac{1}{3(n-2)!} \right)^3 \\
& - \frac{6(n-2)(n-3)}{(n-1)^4} \left( \frac{1}{e} + \frac{1}{(n-2)e} + \frac{1}{3(n-2)!} \right)^2 \\
& \quad \left( \frac{1}{e} + \frac{2}{(n-2)e} + \frac{1}{(n-2)(n-3)e} + \frac{1}{3(n-2)!} \right) \\
& - \frac{6(n-2)(n-4)(n^2-6n+10)}{(n-1)^4(n-3)} \left( \frac{1}{e} + \frac{1}{(n-2)e} + \frac{1}{3(n-2)!} \right) \\
& \quad \left( \frac{1}{e} + \frac{2}{(n-2)e} + \frac{1}{(n-2)(n-3)e} + \frac{1}{3(n-2)!} \right)^2 \\
& - \frac{n^5 - 16n^4 + 103n^3 - 335n^2 + 551n - 362}{(n-1)^4(n-3)} \\
& \quad \left( \frac{1}{e} + \frac{2}{(n-2)e} + \frac{1}{(n-2)(n-3)e} + \frac{1}{3(n-2)!} \right)^3 \\
& = \frac{1}{8(n-1)^4(n-2)(n-3)^4} \\
& \quad \left( \frac{8n^7 - 88n^6 + 360n^5 - 664n^4 + 520n^3 - 168n^2 + 184n - 152}{(n-2)e^3} \right. \\
& \quad - \frac{512n^5 - 5120n^4 + 21504n^3 - 49125n^2 + 62868n - 36271}{1728(n-2)^2(n-4)!^3} \\
& \quad - \frac{64n^6 - 704n^5 + 3127n^4 - 7233n^3 + 9081n^2 - 5103n + 128}{24(n-2)(n-4)!^2e} \\
& \quad \left. - \frac{8n^7 - 93n^6 + 426n^5 - 980n^4 + 1182n^3 - 631n^2 - 8n - 24}{(n-4)!e^2} \right).
\end{aligned}$$

The expression in parentheses is greater than 8831 when  $n = 8$  and, by Lemma 11, increasing in  $n$  when  $n \geq 8$ . Thus, for  $n \geq 8$ ,

$$(\hat{d}_{n-1}^1)^3 - \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \geq \frac{8831}{8(n-1)^4(n-2)(n-3)^4} > \frac{967}{(n-1)^9}.$$

This completes the proof for  $n \geq 8$ .

For  $4 \leq n \leq 7$  we can compute both of the terms in the statement exactly. It is easily verified that  $\hat{d}_3^1 = \frac{3}{6} = \frac{1}{2}$ ,  $\hat{d}_4^1 = \frac{11}{24}$ ,  $\hat{d}_5^1 = \frac{53}{120}$ , and  $\hat{d}_6^1 = \frac{309}{720}$ , which gives us the value of the first term. The value of the second term is given in Lemma 4. Thus, for  $n = 4$ ,

$$(\hat{d}_3^1)^3 - \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] = \frac{1}{8} - \frac{1}{8} = 0.$$

For  $n = 5$ ,

$$(\hat{d}_4^1)^3 - \mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] = \frac{1331}{13824} - \frac{5}{54} = \frac{51}{13824} = \frac{17}{4608} > \frac{967}{4^9}.$$

For  $n = 6$ ,

$$(\hat{d}_5^1)^3 - \mathbb{P}[\bar{R}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = M^3] = \frac{148877}{1728000} - \frac{439471}{5184000} = \frac{179}{129600} > \frac{967}{5^9}.$$

Finally, for  $n = 7$ ,

$$(\hat{d}_6^1)^3 - \mathbb{P}[\bar{R}_{\text{FSD}(\theta)}^{\leq 3} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = M^3] = \frac{1092727}{13824000} - \frac{406417}{5184000} = \frac{5369}{8294400} > \frac{967}{6^9}.$$

This completes the proof.  $\square$

## D Proofs Deferred from Section 4.2

### D.1 Proof of Lemma 7

**Lemma 7.** *For every  $n \in \mathbb{N}$  with  $n \geq 2$  and  $b \in [n-1] \cup \{0\}$ ,*

$$\varphi_b(n) = \frac{1}{1 - \hat{d}_n^b} \left( \frac{(n+1)^2}{n+1-b} (1 - \hat{d}_{n+1}^b) - (n+1) \hat{d}_n^b \right).$$

*Proof.* Fix  $n$  and  $b$  as in the statement. Throughout this proof, the probabilities and expectations are taken with respect to a permutation  $\pi$  taken uniformly at random from  $\mathcal{P}(\{1+b, \dots, n+b\})$ . We let  $p_b(n, k)$  denote the probability that such a permutation has exactly  $k$  fixed points. In general, for  $m \in \mathbb{N}$  with  $m \geq 2$ ,  $c \in [m] \cup \{0\}$ , and  $\ell \in [m-c] \cup \{0\}$ , we define  $p_c(m, \ell) := \mathbb{P}[|\{i \in [m] : \rho(i) = i\}| = \ell]$ , where  $\rho$  is a permutation taken uniformly at random from the set  $\mathcal{P}(\{1+c, \dots, m+c\})$ .

For each  $k \in [n-b] \cup \{0\}$ , we have that

$$\begin{aligned} p_b(n, k) &= \frac{1}{n!} |\{\pi \in \mathcal{P}(\{1+b, \dots, n+b\}) : |\{i \in \{1+b, \dots, n\} : \pi(i) = i\}| = k\}| \\ &= \frac{1}{n!} \sum_{S \subseteq \{1+b, \dots, n\} : |S|=k} |\{\pi \in \mathcal{P}(\{1+b, \dots, n+b\}) : \pi(i) = i \forall i \in S, \pi(i) \neq i \forall i \notin S\}| \\ &= \frac{1}{n!} \binom{n-b}{k} |\{\pi \in \mathcal{P}(\{1+b, \dots, n+b\}) : \\ &\quad \pi(i) = i \forall i \in \{1+b, \dots, k+b\}, \pi(i) \neq i \forall i \in \{k+1+b, \dots, n\}\}| \\ &= \frac{1}{n!} \binom{n-b}{k} |\{\pi \in \mathcal{P}(\{k+1+b, \dots, n+b\}) : \pi(i) \neq i \forall i \in \{k+1+b, \dots, n-k\}\}| \\ &= \frac{1}{n!} \binom{n-b}{k} |\{\pi \in \mathcal{P}(\{1+b, \dots, n-k+b\}) : \pi(i) \neq i \forall i \in \{1+b, \dots, n-k\}\}| \\ &= \frac{1}{n!} \binom{n-b}{k} (n-k)! \cdot p_b(n-k, 0) \\ &= \frac{(n-b)!}{n!} \cdot \frac{(n-k)!}{(n-k-b)!} \cdot \frac{1}{k!} \cdot p_b(n-k, 0), \end{aligned} \tag{13}$$

where the second equality follows by applying total probabilities conditional on the  $k$  fixed points of the permutation, and the third one by observing that the terms of the sum are the same for every set  $S$ , so we can take an arbitrary one and reduce to counting permutations on  $\{k+1+b, \dots, n+b\}$

with no fixed points. The subsequent equalities follow by shifting, simplifying, and applying the definition of  $p_b(n-k, 0)$ .

Since every permutation in  $\mathcal{P}(\{1+b, \dots, n+b\})$  has a number of fixed points between 0 and  $n-b$ , we know that  $\sum_{k=0}^{n-b} p_b(n, k) = 1$ . Therefore, (13) yields

$$p_b(n, 0) = 1 - \sum_{k=1}^{n-b} \frac{(n-b)!}{n!} \cdot \frac{(n-k)!}{(n-k-b)!} \cdot \frac{1}{k!} \cdot p_b(n-k, 0). \quad (14)$$

We now note that, conditional on a permutation of length  $n$  having  $k$  fixed points, the expected index of the first one is simply  $\frac{n+1}{k+1}$ , i.e.,<sup>9</sup>

$$\mathbb{E}[\min\{i \in [n] : \pi(i) = i\} \mid |\{i \in [n] : \pi(i) = i\}| = k] = \frac{n+1}{k+1}. \quad (15)$$

To see this, we observe that the proportion of permutations of length  $n$  whose  $k$  fixed points occur at indices in  $\{j, \dots, n\}$  is  $\binom{n+1-j}{k} / \binom{n}{k}$ . Therefore, the above expectation is equal to

$$\sum_{j=1}^{n-k+1} \frac{\binom{n+1-j}{k}}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{j=k}^n \binom{j}{k} = \frac{1}{\binom{n}{k}} \binom{n+1}{k+1} = \frac{n+1}{k+1},$$

where the third equality follows from the hockey-stick identity.

We can now combine the previous equations to obtain

$$\begin{aligned} \varphi_b(n) &= \frac{\sum_{k=1}^{n-b} \mathbb{E}[\min\{i \in [n] : \pi(i) = i\} \mid |\{i \in [n] : \pi(i) = i\}| = k] \cdot p_b(n, k)}{\mathbb{P}[\{i \in [n] : \pi(i) = i\} \neq \emptyset]} \\ &= \frac{1}{1 - p_b(n, 0)} \sum_{k=1}^{n-b} \frac{n+1}{k+1} \cdot \frac{(n-b)!}{n!} \cdot \frac{(n-k)!}{(n-k-b)!} \cdot \frac{1}{k!} \cdot p_b(n-k, 0) \\ &= \frac{1}{1 - p_b(n, 0)} \cdot \frac{(n+1)^2}{n+1-b} \sum_{k=1}^{n-b} \frac{(n+1-b)!}{(n+1)!} \cdot \frac{(n+1-(k+1))!}{(n+1-(k+1)-b)!} \cdot \frac{1}{(k+1)!} \\ &\quad \cdot p_b(n+1-(k+1), 0) \\ &= \frac{1}{1 - p_b(n, 0)} \cdot \frac{(n+1)^2}{n+1-b} \sum_{k=2}^{n+1-b} \frac{(n+1-b)!}{(n+1)!} \cdot \frac{(n+1-k)!}{(n+1-k-b)!} \cdot \frac{1}{k!} \cdot p_b(n+1-k, 0) \\ &= \frac{1}{1 - p_b(n, 0)} \cdot \frac{(n+1)^2}{n+1-b} \left( 1 - p_b(n+1, 0) - \frac{(n+1-b)!}{(n+1)!} \cdot \frac{n!}{(n-b)!} \cdot p_b(n, 0) \right) \\ &= \frac{1}{1 - p_b(n, 0)} \left( \frac{(n+1)^2}{n+1-b} (1 - p_b(n+1, 0)) - (n+1)p_b(n, 0) \right). \end{aligned}$$

where the second equality follows from (13) and (15) and the fifth one from (14). The result now directly follows from Lemma 1, since  $p_b(n, 0) = d_n^b$  and  $p_b(n+1, 0) = d_{n+1}^b$ .  $\square$

<sup>9</sup>This fact was proven by Anderson and Weber [1990] for the case with  $b = 0$  but does not depend on this value; we provide a proof for completeness.

## D.2 Proof of Lemma 8

**Lemma 8.** For every  $n \in \mathbb{N}$  with  $n \geq 4$ ,  $\theta \in [0, 1)$ , and  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ ,

$$\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,1}] = 2(n-1) + 1 + \frac{1}{1 - \hat{d}_{n-2}^0} (n-1)(1 - \hat{d}_{n-1}^0 - \hat{d}_{n-2}^0),$$

$$\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,2}] = 2(n-1) + 1 + \frac{1}{1 - \hat{d}_{n-2}^1} \left( \frac{(n-1)^2}{n-2} (1 - \hat{d}_{n-1}^1) - (n-1)\hat{d}_{n-2}^1 \right),$$

$$\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,4}] = 2(n-1) + 1.$$

For every  $n \in \mathbb{N}$  with  $n \geq 5$ ,  $\theta \in [0, 1)$ , and  $\sigma \in \{\text{AW}(\theta), \text{FSD}(\theta)\}$ ,

$$\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,3}] = 2(n-1) + 1 + \frac{1}{1 - \hat{d}_{n-2}^2} \left( \frac{(n-1)^2}{n-3} (1 - \hat{d}_{n-1}^2) - (n-1)\hat{d}_{n-2}^2 \right).$$

*Proof.* We fix  $n$ ,  $\theta$ , and  $\sigma$  as in the statement. The fourth equality is trivial, since the events  $\chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3$  and  $B_{\sigma,4}$  jointly imply that the players meet in the first step of the third round, i.e.,  $\mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,4}] = 2(n-1) + 1$ .

For the other expressions, we apply Lemma 7. Conditional on  $\mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3$ , and  $B_{\sigma,1}$ , both players visit the other's home location at the beginning of the third round, so the expected meeting time, after the first  $2(n-1) + 1$  steps, is the expected minimum index for which two permutations taken uniformly at random from  $\mathcal{P}(\{3, \dots, n\})$  coincide, conditional on them coinciding. This is the same as the expected index of the first fixed point of a permutation  $\pi \in \mathcal{P}([n-2])$  taken uniformly at random, conditional on having at least one fixed point, i.e.,  $\varphi_0(n-2)$ . Thus, we obtain from Lemma 7 that

$$\begin{aligned} \mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,1}] &= 2(n-1) + 1 + \varphi_0(n-2) \\ &= 2(n-1) + 1 + \frac{1}{1 - \hat{d}_{n-2}^0} \left( \frac{(n-1)^2}{n-1} (1 - \hat{d}_{n-1}^0) - (n-1)\hat{d}_{n-2}^0 \right) \\ &= 2(n-1) + 1 + \frac{1}{1 - \hat{d}_{n-2}^0} (n-1)(1 - \hat{d}_{n-1}^0 - \hat{d}_{n-2}^0). \end{aligned}$$

Conditional on  $\mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3$ , and  $B_{\sigma,2}$ , exactly one player (say player 1 w.l.o.g.) visits the other's home location at the beginning of the third round, so the expected meeting time, after the first  $2(n-1) + 1$  steps, is the expected minimum index for which a permutation taken uniformly at random from  $\mathcal{P}(\{3, \dots, n\})$  and a permutation taken uniformly at random from  $\mathcal{P}([n] \setminus \{2, j\})$  for some  $j \in [n] \setminus \{1, 2\}$  coincide, conditional on them coinciding. This is the same as the expected index of the first fixed point of a permutation  $\pi \in \mathcal{P}(\{2, \dots, n-1\})$  taken uniformly at random, conditional on having at least one fixed point, i.e.,  $\varphi_1(n-2)$ . Thus, we obtain from Lemma 7 that

$$\begin{aligned} \mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,2}] &= 2(n-1) + 1 + \varphi_1(n-2) \\ &= 2(n-1) + 1 + \frac{1}{1 - \hat{d}_{n-2}^1} \left( \frac{(n-1)^2}{n-2} (1 - \hat{d}_{n-1}^1) - (n-1)\hat{d}_{n-2}^1 \right). \end{aligned}$$

Conditional on  $\mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3$ , and  $B_{\sigma,3}$ , no player visits the other's home location at the beginning of the third round, but the visited locations are not the same; note that meeting in this

round is only possible if  $n \geq 5$ . Therefore, the expected meeting time, after the first  $2(n-1)+1$  steps, is the expected minimum index for which a permutation taken uniformly at random from  $\mathcal{P}([n] \setminus \{1, j\})$  and a permutation taken uniformly at random from  $\mathcal{P}([n] \setminus \{2, k\})$  for  $j, k \in [n] \setminus \{1, 2\}$  with  $j \neq k$  coincide, conditional on them coinciding. This is the same as the expected index of the first fixed point of a permutation  $\pi \in \mathcal{P}(\{3, \dots, n\})$  taken uniformly at random, conditional on having at least one fixed point, i.e.,  $\varphi_2(n-2)$ . Thus, we obtain from Lemma 7 that

$$\begin{aligned} \mathbb{E}[t_\sigma \mid \mathbf{R}_\sigma^3, \chi_\sigma^1 = \chi_\sigma^2 = \mathbf{M}^3, B_{\sigma,2}] &= 2(n-1) + 1 + \varphi_2(n-2) \\ &= 2(n-1) + 1 + \frac{1}{1 - \hat{d}_{n-2}^2} \left( \frac{(n-1)^2}{n-3} (1 - \hat{d}_{n-1}^2) - (n-1) \hat{d}_{n-2}^2 \right). \square \end{aligned}$$

### D.3 Proof of Lemma 9

**Lemma 9.** *For every  $n \in \mathbb{N}$  with  $n \geq 4$  and  $\theta \in [0, 1)$ ,*

$$\begin{aligned} \mathbb{P}[B_{\text{FSD}(\theta),1}, \bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 2} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ = \frac{1}{(n-1)^4} \left( (\hat{d}_{n-2}^0)^2 + 2(n-2)(\hat{d}_{n-2}^1)^2 + \frac{(n-2)(n^2 - 7n + 13)}{n-3} (\hat{d}_{n-2}^2)^2 \right), \end{aligned}$$

$$\begin{aligned} \mathbb{P}[B_{\text{FSD}(\theta),2}, \bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 2} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ = \frac{2(n-2)}{(n-1)^4} \left( 2\hat{d}_{n-2}^0 \hat{d}_{n-2}^1 + (n-3)(\hat{d}_{n-2}^1)^2 + 2(n-3)\hat{d}_{n-2}^1 \hat{d}_{n-2}^2 + \frac{(n-4)(n^2 - 6n + 10)}{n-3} (\hat{d}_{n-2}^2)^2 \right), \end{aligned}$$

$$\begin{aligned} \mathbb{P}[B_{\text{FSD}(\theta),3}, \bar{\mathbf{R}}_{\text{FSD}(\theta)}^{\leq 2} \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ = \frac{(n-2)}{(n-1)^4} \left( \frac{2(n^2 - 7n + 13)}{n-3} \hat{d}_{n-2}^0 \hat{d}_{n-2}^2 + 2(n-3)(\hat{d}_{n-2}^1)^2 + \frac{4(n-4)(n^2 - 6n + 10)}{n-3} \hat{d}_{n-2}^1 \hat{d}_{n-2}^2 \right. \\ \left. + \frac{n^4 - 14n^3 + 75n^2 - 185n + 181}{n-3} (\hat{d}_{n-2}^2)^2 \right). \end{aligned}$$

*Proof.* Let  $n$  and  $\theta$  be as in the statement. Since the statement is only about the strategy  $\text{FSD}(\theta)$ , we omit the subindex  $\text{FSD}(\theta)$  throughout the proof for compactness.

We first compute  $\mathbb{P}[B_1, \bar{\mathbf{R}}^{\leq 2} \mid \chi^1 = \chi^2 = \mathbf{M}^3]$ . To do so, we first apply the law of total probability to obtain

$$\begin{aligned} \mathbb{P}[B_1, \bar{\mathbf{R}}^{\leq 2} \mid \chi^1 = \chi^2 = \mathbf{M}^3] &= \sum_{k,\ell \in [4]} \mathbb{P}[\bar{\mathbf{R}}^{\leq 2} \mid \chi^1 = \chi^2 = \mathbf{M}^3, B_1, A_k^1, A_\ell^2] \\ &\quad \cdot \mathbb{P}[B_1 \mid \chi^1 = \chi^2 = \mathbf{M}^3, A_k^1, A_\ell^2] \cdot \mathbb{P}[A_k^1, A_\ell^2 \mid \chi^1 = \chi^2 = \mathbf{M}^3]. \end{aligned}$$

We now observe that, for  $i \in \{1, 2\}$ ,  $\pi_3^i(1) = 3 - i$  occurs with probability 1 conditional on  $\chi^i = \mathbf{M}^3$  and  $A_1^i$  (because  $i$  chooses  $3 - i$  as the first location in the first three rounds), with probability  $1/3$  conditional on  $\chi^i = \mathbf{M}^3$  and  $A_3^i$  (because  $i$  chooses  $3 - i$  as the first location in one of the three rounds), and with probability 0 conditional on  $\chi^i = \mathbf{M}^3$  and either  $A_2^i$  or  $A_4^i$ . Regarding the events  $A_k^1$  and  $A_\ell^2$ , we note that when we only condition on  $\chi^1 = \chi^2 = \mathbf{M}^3$  they are independent and, from Lemma 3,  $\mathbb{P}[A_1^i \mid \chi^1 = \chi^2 = \mathbf{M}^3] = \frac{1}{(n-1)^2}$  and  $\mathbb{P}[A_3^i \mid \chi^1 = \chi^2 = \mathbf{M}^3] = \frac{3(n-2)}{(n-1)^2}$  for  $i \in \{1, 2\}$ .

Thus,

$$\begin{aligned}
& \mathbb{P}[B_1, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3] \\
&= \frac{1}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_1, A_1^1, A_1^2] \\
&\quad + 2 \cdot \frac{1}{3} \cdot \frac{3(n-2)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_1, A_1^1, A_3^2] \\
&\quad + \frac{1}{9} \cdot \frac{9(n-2)^2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_1, A_3^1, A_3^2] \\
&= \frac{1}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_1, A_1^1, A_1^2] + \frac{2(n-2)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_1, A_1^1, A_3^2] \\
&\quad + \frac{(n-2)^2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_1, A_3^1, A_3^2].
\end{aligned}$$

To compute the remaining probabilities, we apply Lemma 1. The expression  $\mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_1, A_1^1, A_1^2]$  is the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_r^i(1) = 3 - i$  for each  $i \in \{1, 2\}$  and  $r \in \{1, 2\}$ . This probability is  $\hat{d}_{n-2}^0$  for each round, hence  $(\hat{d}_{n-2}^0)^2$  for both. Similarly,  $\mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_1, A_1^1, A_3^2]$  is the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_r^1(1) = 2$  and  $\pi_r^2(1) \neq 1$  for  $r \in \{1, 2\}$ . This probability is  $\hat{d}_{n-2}^1$  for each round, hence  $(\hat{d}_{n-2}^1)^2$  for both. Finally,  $\mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_1, A_3^1, A_3^2]$  is the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_r^i(1) \neq 3 - i$  for  $i \in \{1, 2\}$  and  $r \in \{1, 2\}$ , and  $\pi_1^1(1) \neq \pi_2^1(1)$  for  $i \in \{1, 2\}$ . Conditional on these events, the probability of meeting in the first step of round 1 is  $\frac{1}{n-2}$ , and the probability of not meeting in the first step of round 1 and meeting in the first step of round 2 is  $\frac{n-4}{n-2} \cdot \frac{1}{n-3}$ . The non-meeting probability in later steps of these rounds is  $\hat{d}_{n-2}^2$  for each round, hence  $(\hat{d}_{n-2}^2)^2$  for both. Therefore, the overall non-meeting probability in both rounds is  $(1 - \frac{1}{n-2} - \frac{n-4}{n-2} \cdot \frac{1}{n-3})(\hat{d}_{n-2}^2)^2 = \frac{n^2-7n+13}{(n-2)(n-3)}(\hat{d}_{n-2}^2)^2$ . We obtain

$$\mathbb{P}[B_1, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3] = \frac{1}{(n-1)^4} \left( (\hat{d}_{n-2}^0)^2 + 2(n-2)(\hat{d}_{n-2}^1)^2 + \frac{(n-2)(n^2-7n+13)}{n-3} (\hat{d}_{n-2}^2)^2 \right).$$

which concludes the proof for  $\mathbb{P}[B_1, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3]$ .

We now proceed similarly to compute  $\mathbb{P}[B_2, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3]$ . We first note that, because of symmetry, this probability is two times  $\mathbb{P}[\pi_3^1(1) = 2, \pi_3^2(1) \neq 1, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3]$ , so we can apply total probabilities to obtain

$$\begin{aligned}
& \mathbb{P}[B_2, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3] = \\
& \quad 2 \sum_{k, \ell \in [4]} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_k^1, A_\ell^2] \\
& \quad \cdot \mathbb{P}[\pi_3^1(1) = 2, \pi_3^2(1) \neq 1 \mid \chi^1 = \chi^2 = M^3, A_k^1, A_\ell^2] \mathbb{P}[A_k^1, A_\ell^2 \mid \chi^1 = \chi^2 = M^3].
\end{aligned}$$

We know from the previous case that  $\pi_3^1(1) = 2$  occurs with probability 1 conditional on  $\chi^1 = M^3$  and  $A_1^1$ , with probability  $1/3$  conditional on  $\chi^1 = M^3$  and  $A_3^1$ , and with probability 0 conditional

on  $\chi^1 = M^3$  and either  $A_2^1$  or  $A_4^1$ . On the other hand,  $\pi_3^2(1) \neq 1$  occurs with probability 1 conditional on  $\chi^1 = M^3$  and  $A_2^2$  or  $A_4^2$ , with probability  $2/3$  conditional on  $\chi^1 = M^3$  and  $A_3^2$ , and with probability 0 conditional on  $\chi^1 = M^3$  and  $A_1^2$ . Regarding the events  $A_k^1$  and  $A_\ell^2$ , we note that when we only condition on  $\chi^1 = \chi^2 = M^3$  they are independent and, from Lemma 3,  $\mathbb{P}[A_1^i | \chi^1 = \chi^2 = M^3] = \frac{1}{(n-1)^2}$ ,  $\mathbb{P}[A_2^i | \chi^1 = \chi^2 = M^3] = \frac{n-2}{(n-1)^2}$ ,  $\mathbb{P}[A_3^i | \chi^1 = \chi^2 = M^3] = \frac{3(n-2)}{(n-1)^2}$ , and  $\mathbb{P}[A_4^i | \chi^1 = \chi^2 = M^3] = \frac{(n-2)(n-4)}{(n-1)^2}$  for  $i \in \{1, 2\}$ . Thus,

$$\begin{aligned}
& \mathbb{P}[B_2, \bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3] \\
&= 2 \cdot \frac{n-2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_1^1, A_2^2] \\
&+ 2 \cdot \frac{2}{3} \cdot \frac{3(n-2)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_1^1, A_3^2] \\
&+ 2 \cdot \frac{(n-2)(n-4)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_1^1, A_4^2] \\
&+ 2 \cdot \frac{1}{3} \cdot \frac{3(n-2)^2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_3^1, A_2^2] \\
&+ 2 \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{9(n-2)^2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_3^1, A_3^2] \\
&+ 2 \cdot \frac{1}{3} \cdot \frac{3(n-2)^2(n-4)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_3^1, A_4^2] \\
&= \frac{2(n-2)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_1^1, A_2^2] \\
&+ \frac{4(n-2)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_1^1, A_3^2] \\
&+ \frac{2(n-2)(n-4)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_1^1, A_4^2] \\
&+ \frac{2(n-2)^2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_3^1, A_2^2] \\
&+ \frac{4(n-2)^2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_3^1, A_3^2] \\
&+ \frac{2(n-2)^2(n-4)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_3^1, A_4^2].
\end{aligned}$$

For  $k \in \{1, 3\}$  and  $\ell \in \{2, 3, 4\}$ , we write

$$p_{k\ell} := \mathbb{P}[\bar{R}^{\leq 2} | \chi^1 = \chi^2 = M^3, \pi_3^1(1) = 2, \pi_3^2(1) \neq 1, A_k^1, A_\ell^2]$$

in what follows, for the sake of compactness. We compute these probabilities by applying Lemma 1.

Both  $p_{12}$  and  $p_{14}$  are equal to the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_r^1(1) = 2$  and  $\pi_r^2(1) \neq 1$  for each  $r \in \{1, 2\}$ . This probability is  $\hat{d}_{n-2}^1$  for each round, hence  $(\hat{d}_{n-2}^1)^2$  for both. Similarly,  $p_{13}$  is equal to the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_r^1(1) = 2$  for each  $r \in \{1, 2\}$ ,  $\pi_r^2(1) = 1$  for some fixed  $r \in \{1, 2\}$ , and  $\pi_s^2(1) \neq 1$  for  $s \in \{1, 2\} \setminus \{r\}$ . This

probability is  $\hat{d}_{n-2}^0$  for round  $r$  and  $\hat{d}_{n-2}^1$  for round  $s$ , hence  $\hat{d}_{n-2}^0\hat{d}_{n-2}^1$  for both. The probability  $p_{32}$  corresponds to the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_r^i(1) \neq 3-i$  for  $i \in \{1, 2\}$  and  $r \in \{1, 2\}$ ,  $\pi_1^1(1) \neq \pi_2^1(1)$ , and  $\pi_1^2(1) = \pi_2^2(1)$ . Conditional on these events, the probability of not meeting in the first step of rounds 1 and 2 is  $\frac{n-4}{n-2}$ . Conditional on not meeting in the first step, the non-meeting probability in later steps of these rounds is  $\hat{d}_{n-2}^2$  for each round, hence  $(\hat{d}_{n-2}^2)^2$  for both. The probability  $p_{33}$  corresponds to the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_1^i(1) \neq \pi_2^i(1)$  for  $i \in \{1, 2\}$ ,  $2 \notin \{\pi_1^1(1), \pi_2^1(1)\}$ , and  $1 \in \{\pi_1^2(1), \pi_2^2(1)\}$ . Conditional on these events, the probability of not meeting in the first step of rounds 1 and 2 is  $\frac{n-3}{n-2}$ . Conditional on not meeting in the first step, the non-meeting probability in later steps of these rounds is  $\hat{d}_{n-2}^1$  for the round  $r$  with  $\pi_r^2(1) = 1$  and  $\hat{d}_{n-2}^2$  for the other round in  $\{1, 2\} \setminus \{r\}$ , hence  $\hat{d}_{n-2}^1\hat{d}_{n-2}^2$  for both. Finally,  $p_{34}$  is the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_r^i(1) \neq 3-i$  for  $i \in \{1, 2\}$  and  $r \in \{1, 2\}$ , and  $\pi_1^i(1) \neq \pi_2^i(1)$  for  $i \in \{1, 2\}$ . This probability was already computed in the previous case, where we obtained that the non-meeting probability in both rounds is  $\frac{n^2-7n+13}{(n-2)(n-3)}(\hat{d}_{n-2}^2)^2$ . We obtain

$$\begin{aligned} & \mathbb{P}[B_2, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3] \\ &= \frac{2(n-2)}{(n-1)^4} \left( (\hat{d}_{n-2}^1)^2 + 2\hat{d}_{n-2}^0\hat{d}_{n-2}^1 + (n-4)(\hat{d}_{n-2}^1)^2 + (n-2)\frac{n-4}{n-2}(\hat{d}_{n-2}^2)^2 \right. \\ & \quad \left. + 2(n-2)\frac{n-3}{n-2}\hat{d}_{n-2}^1\hat{d}_{n-2}^2 + (n-2)(n-4)\frac{n^2-7n+13}{(n-2)(n-3)}(\hat{d}_{n-2}^2)^2 \right) \\ &= \frac{2(n-2)}{(n-1)^4} \left( 2\hat{d}_{n-2}^0\hat{d}_{n-2}^1 + (n-3)(\hat{d}_{n-2}^1)^2 + 2(n-3)\hat{d}_{n-2}^1\hat{d}_{n-2}^2 + \frac{(n-4)(n^2-6n+10)}{n-3}(\hat{d}_{n-2}^2)^2 \right), \end{aligned}$$

which concludes the proof for  $\mathbb{P}[B_2, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3]$ .

We finally compute  $\mathbb{P}[B_3, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3]$ . As before, we first apply total probabilities to obtain

$$\begin{aligned} \mathbb{P}[B_3, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3] &= \sum_{k, \ell \in [4]} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_k^1, A_\ell^2] \\ & \quad \cdot \mathbb{P}[B_3 \mid \chi^1 = \chi^2 = M^3, A_k^1, A_\ell^2] \mathbb{P}[A_k^1, A_\ell^2 \mid \chi^1 = \chi^2 = M^3]. \end{aligned}$$

We next compute the probability of  $B_3$  conditional on  $\chi^1 = \chi^2 = M^3$ ,  $A_k^1$ , and  $A_\ell^2$  for different values of  $k, \ell \in [4]$ ; we restrict to cases with  $k \leq \ell$  because of symmetry. If  $k = 1$  this probability is 0, because player 1 chooses 2 as the first location in the first three rounds and  $B_3$  cannot hold. If  $k, \ell \in \{2, 4\}$ , it is  $\frac{n-3}{n-2}$ , because  $B_3$  holds whenever the starting locations chosen by each player in the third round differ. If  $k = \ell = 3$ , it is  $(\frac{2}{3})^2 \frac{n-3}{n-2}$ , because for  $B_3$  to hold we need that  $\pi_r^i(1) = 3-i$  for a round  $r \in \{1, 2\}$  for  $i \in \{1, 2\}$ , which occurs with probability  $2/3$  for each player, and that  $\pi_3^1(1) \neq \pi_3^2(1)$ , which occurs with probability  $\frac{n-3}{n-2}$  conditional on the previous event. If  $k = 2$  and  $\ell = 3$ , it is  $\frac{2}{3} \cdot \frac{n-3}{n-2}$ , because for  $B_3$  to hold we need that  $\pi_r^2(1) = 1$  for a round  $r \in \{1, 2\}$ , which occurs with probability  $2/3$ , and that  $\pi_3^1(1) \neq \pi_3^2(1)$ , which occurs with probability  $\frac{n-3}{n-2}$  conditional on the previous event. Finally, if  $k = 3$  and  $\ell = 4$ , it is  $\frac{2}{3} \cdot \frac{n-3}{n-2}$ , because for  $B_3$  to hold we need that  $\pi_r^1(1) = 2$  for a round  $r \in \{1, 2\}$ , which occurs with probability  $2/3$ ,

and that  $\pi_3^1(1) \neq \pi_3^2(1)$ , which occurs with probability  $\frac{n-3}{n-2}$  conditional on the previous event. Regarding the events  $A_k^1$  and  $A_\ell^2$ , we note that when we only condition on  $\chi^1 = \chi^2 = M^3$  they are independent and, from Lemma 3,  $\mathbb{P}[A_2^i \mid \chi^1 = \chi^2 = M^3] = \frac{n-2}{(n-1)^2}$ ,  $\mathbb{P}[A_3^i \mid \chi^1 = \chi^2 = M^3] = \frac{3(n-2)}{(n-1)^2}$ , and  $\mathbb{P}[A_4^i \mid \chi^1 = \chi^2 = M^3] = \frac{(n-2)(n-4)}{(n-1)^2}$  for  $i \in \{1, 2\}$ . Thus,

$$\begin{aligned}
\mathbb{P}[B_3, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3] &= \frac{n-3}{n-2} \cdot \frac{(n-2)^2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_2^1, A_2^2] \\
&+ 2 \cdot \frac{2}{3} \cdot \frac{n-3}{n-2} \cdot \frac{3(n-2)^2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_2^1, A_3^2] \\
&+ 2 \cdot \frac{n-3}{n-2} \cdot \frac{(n-2)^2(n-4)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_2^1, A_4^2] \\
&+ \frac{4}{9} \cdot \frac{n-3}{n-2} \cdot \frac{9(n-2)^2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_3^1, A_3^2] \\
&+ 2 \cdot \frac{2}{3} \cdot \frac{n-3}{n-2} \cdot \frac{3(n-2)^2(n-4)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_3^1, A_4^2] \\
&+ \frac{n-3}{n-2} \cdot \frac{(n-2)^2(n-4)^2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_4^1, A_4^2] \\
&= \frac{(n-2)(n-3)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_2^1, A_2^2] \\
&+ \frac{4(n-2)(n-3)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_2^1, A_3^2] \\
&+ \frac{2(n-2)(n-3)(n-4)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_2^1, A_4^2] \\
&+ \frac{4(n-2)(n-3)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_3^1, A_3^2] \\
&+ \frac{4(n-2)(n-3)(n-4)}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_3^1, A_4^2] \\
&+ \frac{(n-2)(n-3)(n-4)^2}{(n-1)^4} \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_4^1, A_4^2].
\end{aligned}$$

For  $k, \ell \in \{2, 3, 4\}$  with  $k \leq \ell$ , we write

$$p_{k\ell} := \mathbb{P}[\bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3, B_3, A_k^1, A_\ell^2]$$

in what follows, for the sake of compactness. We compute these probabilities by applying Lemma 1.

The probability  $p_{22}$  is the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $2 \neq \pi_r^1(1) \neq \pi_r^2(1) \neq 1$  for each  $r \in \{1, 2\}$ , because we know that  $\pi_1^1(1) = \pi_2^1(1) = \pi_3^1(1)$  and  $\pi_1^2(1) = \pi_2^2(1) = \pi_3^2(1)$  and that  $2 \neq \pi_3^1(1) \neq \pi_3^2(1) \neq 1$ . This probability is  $\hat{d}_{n-2}^2$  for each round, hence  $(\hat{d}_{n-2}^2)^2$  for both.

The probability  $p_{23}$  is the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_1^1(1) = \pi_2^1(1) = \pi_3^1(1) \neq 2$  and  $\pi_r^2(1)$  being different for all  $r \in [3]$  with  $\pi_3^1(1) \neq \pi_3^2(1)$  and  $\pi_r^2(1) = 1$  for some  $r \in \{1, 2\}$ . The probability that the players do not meet in the first step of rounds 1 and 2 is then  $\frac{n-4}{n-3}$ . Conditional on this event, the non-meeting

probability in later steps of these rounds is  $\hat{d}_{n-2}^1$  for the round  $r \in \{1, 2\}$  with  $\pi_r^2(1) = 1$  and  $\hat{d}_{n-2}^2$  for the other round in  $\{1, 2\} \setminus \{r\}$ , hence  $\hat{d}_{n-2}^1 \hat{d}_{n-2}^2$  for both.

The probability  $p_{24}$  is the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_1^1(1) = \pi_2^1(1) = \pi_3^1(1) \neq 2$  and  $\pi_r^2(1)$  being different and not equal to 1 for all  $r \in [3]$ , with  $\pi_3^1(1) \neq \pi_3^2(1)$ . The probability that the players do not meet in the first step of rounds 1 and 2 is then  $\frac{n-5}{n-3}$ . Conditional on this event, the non-meeting probability in later steps of these rounds is  $\hat{d}_{n-2}^2$  for each round  $r \in \{1, 2\}$ , hence  $(\hat{d}_{n-2}^2)^2$  for both.

The probability  $p_{33}$  is the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_r^1(1)$  being different for all  $r \in [3]$  with  $\pi_r^1(1) = 2$  for a fixed  $r \in [2]$ ,  $\pi_{r'}^2(1)$  being different for all  $r' \in [3]$  with  $\pi_{r'}^2(1) = 1$  for a fixed  $r' \in [2]$ , and  $\pi_3^1(1) \neq \pi_3^2(1)$ . We distinguish two cases, depending on whether  $r = r'$  or not. If  $r \neq r'$ , which occurs with probability  $1/2$  conditional on the previous events, the players do not meet in the first step of rounds 1 and 2, and the non-meeting probability in later steps of these rounds is  $\hat{d}_{n-2}^1$  for each round, hence  $(\hat{d}_{n-2}^1)^2$  for both. If  $r = r'$ , which occurs with probability  $1/2$  conditional on the previous events as well, then denoting the other round by  $s \in [2] \setminus \{r\}$ , the players do not meet in the first step of rounds 1 or 2 if either  $\pi_3^2(1) = \pi_s^1(1)$ , which occurs with a conditional probability of  $1/(n-3)$ , or  $\pi_3^2(1) \neq \pi_s^1(1)$  and  $\pi_s^2(1) \neq \pi_s^1(1)$ , which occurs with a conditional probability of  $(\frac{n-4}{n-3})^2$ . The probability that the players do not meet in the first step of rounds 1 and 2 is then  $\frac{1}{n-3} + (\frac{n-4}{n-3})^2 = \frac{n^2-7n+13}{(n-3)^2}$ . Conditional on this event, the non-meeting probability in later steps is  $\hat{d}_{n-2}^0$  for round  $r$  and  $\hat{d}_{n-2}^2$  for round  $s$ , hence  $\hat{d}_{n-2}^0 \hat{d}_{n-2}^2$  for both.

The probability  $p_{34}$  is the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_r^1(1)$  being different for all  $r \in [3]$  with  $\pi_r^1(1) = 2$  for some  $r \in [2]$ ,  $\pi_r^2(1)$  being different and not equal to 1 for all  $r \in [3]$ , and  $\pi_3^1(1) \neq \pi_3^2(1)$ . Letting  $r \in [2]$  denote the round such that  $\pi_r^1(1) = 2$  and  $s \in [2] \setminus \{r\}$  the other round, the players do not meet in the first step of rounds 1 and 2 if either  $\pi_3^2(1) = \pi_s^1(1)$ , which occurs with a conditional probability of  $1/(n-3)$ , or  $\pi_3^2(1) \neq \pi_s^1(1)$  and  $\pi_s^2(1) \neq \pi_s^1(1)$ , which occurs with a conditional probability of  $(\frac{n-4}{n-3})^2$ . The probability that the players do not meet in the first step of rounds 1 and 2 is then  $\frac{1}{n-3} + (\frac{n-4}{n-3})^2 = \frac{n^2-7n+13}{(n-3)^2}$ . Conditional on this event, the non-meeting probability in later steps is  $\hat{d}_{n-2}^1$  for round  $r$  and  $\hat{d}_{n-2}^2$  for round  $s$ , hence  $\hat{d}_{n-2}^1 \hat{d}_{n-2}^2$  for both.

Finally, the probability  $p_{44}$  is the probability that the players do not meet in rounds 1 and 2 conditional on moving in these rounds with  $\pi_r^i(1)$  being different and not equal to  $3-i$  for each  $i \in \{1, 2\}$  and  $r \in [3]$ , and  $\pi_3^1(1) \neq \pi_3^2(1)$ . We distinguish two cases, depending on whether  $\pi_3^2(1) \in \{\pi_1^1(1), \pi_2^1(1)\}$  or not. If  $\pi_3^2(1) = \pi_r^1(1)$  for some  $r \in [2]$ , which occurs with probability  $\frac{2}{n-3}$  conditional on the previous events, denoting the other round by  $s \in [2] \setminus \{r\}$ , the players do not meet in the first step of rounds 1 and 2 if  $\pi_s^2(1) \neq \pi_s^1(1)$ , which occurs with a conditional probability of  $\frac{n-4}{n-3}$ . If  $\pi_3^2(1) \notin \{\pi_1^1(1), \pi_2^1(1)\}$ , which occurs with probability  $\frac{n-5}{n-3}$  conditional on the previous events, the players do not meet in the first step of rounds 1 and 2 if either  $\pi_1^2(1) = \pi_2^1(1)$ , which occurs with probability  $\frac{1}{n-3}$ , or if  $\pi_1^2(1) \notin \{\pi_1^1(1), \pi_2^1(1)\}$  and  $\pi_2^2(1) \neq \pi_2^1(1)$ , which occurs with probability  $\frac{n-5}{n-3} \cdot \frac{n-5}{n-4}$ . Therefore, the non-meeting probability in the first step of these rounds is  $\frac{1}{n-3} + \frac{(n-5)^2}{(n-3)(n-4)} = \frac{n^2-9n+21}{(n-3)(n-4)}$ . In either case, conditional on not meeting in the first steps, the non-meeting probability in later steps of these rounds is  $\hat{d}_{n-2}^2$  for each round  $r \in \{1, 2\}$ , hence  $(\hat{d}_{n-2}^2)^2$  for both.

We obtain

$$\begin{aligned}
& \mathbb{P}[B_3, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3] \\
&= \frac{(n-2)(n-3)}{(n-1)^4} \left( (\hat{d}_{n-2}^2)^2 + 4 \cdot \frac{n-4}{n-3} \hat{d}_{n-2}^1 \hat{d}_{n-2}^2 + 2(n-4) \frac{n-5}{n-3} (\hat{d}_{n-2}^2)^2 \right. \\
&\quad \left. + 4 \left( \frac{1}{2} (\hat{d}_{n-2}^1)^2 + \frac{1}{2} \cdot \frac{n^2-7n+13}{(n-3)^2} \hat{d}_{n-2}^0 \hat{d}_{n-2}^2 \right) + 4(n-4) \frac{n^2-7n+13}{(n-3)^2} \hat{d}_{n-2}^1 \hat{d}_{n-2}^2 \right. \\
&\quad \left. + (n-4)^2 \left( \frac{2}{n-3} \cdot \frac{n-4}{n-3} + \frac{n-5}{n-3} \cdot \frac{n^2-9n+21}{(n-3)(n-4)} \right) (\hat{d}_{n-2}^2)^2 \right) \\
&= \frac{(n-2)}{(n-1)^4} \left( \frac{2(n^2-7n+13)}{n-3} \hat{d}_{n-2}^0 \hat{d}_{n-2}^2 + 2(n-3) (\hat{d}_{n-2}^1)^2 + \frac{4(n-4)(n^2-6n+10)}{n-3} \hat{d}_{n-2}^1 \hat{d}_{n-2}^2 \right. \\
&\quad \left. + \frac{n^4-14n^3+75n^2-185n+181}{n-3} (\hat{d}_{n-2}^2)^2 \right),
\end{aligned}$$

which concludes the proof for  $\mathbb{P}[B_3, \bar{R}^{\leq 2} \mid \chi^1 = \chi^2 = M^3]$ .  $\square$

#### D.4 Proof of Lemma 10

**Lemma 10.** *If  $n = 4$ , then  $\Delta_2 = -2\Delta_1 = 8/81$ . If  $n \geq 5$ , then  $\Delta_2 = -2\Delta_1 > 0$  and  $\Delta_3 > 0$ .*

*Proof.* We first express  $d_{n-1}^1$  and  $d_{n-2}^k$  for  $k \in \{0, 1, 2\}$  in terms of  $d_{n-3}^0$  and  $d_{n-4}^0$ , which will be useful to only work with these two terms in what follows. By repeatedly applying the formula  $d_m^{k+1} = d_m^k + d_{m-1}^k$  for  $m \in \mathbb{N}, k \in [m-1]$  from the definition of the difference table, and the well-known recursion  $d_m^0 = (m-1)(d_{m-1}^0 + d_{m-2}^0)$  for  $m \in \mathbb{N} \setminus \{1\}$ , we obtain

$$\begin{aligned}
d_{n-1}^1 &= d_{n-1}^0 + d_{n-2}^0 = (n-2)(d_{n-2}^0 + d_{n-3}^0) + (n-3)(d_{n-3}^0 + d_{n-4}^0) \\
&= (n-2)(n-3)(d_{n-3}^0 + d_{n-4}^0) + (n-2)d_{n-3}^0 + (n-3)(d_{n-3}^0 + d_{n-4}^0) \\
&= (n^2-3n+1)d_{n-3}^0 + (n-1)(n-3)d_{n-4}^0, \\
d_{n-2}^0 &= (n-3)(d_{n-3}^0 + d_{n-4}^0), \\
d_{n-2}^1 &= d_{n-2}^0 + d_{n-3}^0 = (n-3)(d_{n-3}^0 + d_{n-4}^0) + d_{n-3}^0 = (n-2)d_{n-3}^0 + (n-3)d_{n-4}^0, \\
d_{n-2}^2 &= d_{n-2}^1 + d_{n-3}^1 = (n-2)d_{n-3}^0 + (n-3)d_{n-4}^0 + d_{n-3}^0 + d_{n-4}^0 \\
&= (n-1)d_{n-3}^0 + (n-2)d_{n-4}^0.
\end{aligned} \tag{16}$$

We now show that  $\Delta_2 = -2\Delta_1 > 0$ , with  $\Delta_2 = 8/81$  if  $n = 4$ . We first analyze  $\Delta_1$  by observing that

$$\mathbb{P}[B_{AW(\theta),1} \mid \bar{R}_{AW}^{\leq 2}(\theta), \chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3] = \mathbb{P}[B_{AW(\theta),1} \mid \chi_{AW(\theta)}^1 = \chi_{AW(\theta)}^2 = M^3] = \frac{1}{(n-1)^2},$$

because the event  $B_{AW(\theta),1}$  is independent from  $\bar{R}_{AW}^{\leq 2}(\theta)$  and each player starts the third permutation at the home location of the other with probability  $1/(n-1)$ , and

$$\begin{aligned}
& \mathbb{P}[B_{FSD(\theta),1} \mid \bar{R}_{FSD}^{\leq 2}(\theta), \chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3] \\
&= \frac{\mathbb{P}[B_{FSD(\theta),1}, \bar{R}_{FSD}^{\leq 2}(\theta) \mid \chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3]}{\mathbb{P}[\bar{R}_{FSD}^{\leq 2}(\theta) \mid \chi_{FSD(\theta)}^1 = \chi_{FSD(\theta)}^2 = M^3]}
\end{aligned}$$

$$= \frac{(n-1)!^2 \mathbb{P}[B_{\text{FSD}(\theta),1}, \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]}{(d_{n-1}^1)^2},$$

because  $\text{FSD}(\theta)$  is equivalent to  $\text{AW}(\theta)$  in the first two rounds and thus the non-meeting probability in these rounds, when both players move, is simply  $(\hat{d}_{n-1}^1)^2 = (d_{n-1}^1)^2 / (n-1)!^2$ . We obtain

$$\begin{aligned} \Delta_1 &= \mathbb{P}[B_{\text{AW}(\theta),1} \mid \bar{\mathbf{R}}_{\text{AW}}^{\leq 2}(\theta), \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] \\ &\quad - \mathbb{P}[B_{\text{FSD}(\theta),1} \mid \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta), \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ &= \frac{(d_{n-1}^1)^2 - (n-1)^2 (n-1)!^2 \mathbb{P}[B_{\text{FSD}(\theta),1}, \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]}{(n-1)^2 (d_{n-1}^1)^2}. \end{aligned} \quad (17)$$

We now compute (the additive inverse of) the numerator:

$$\begin{aligned} &(n-1)^2 (n-1)!^2 \mathbb{P}[B_{\text{FSD}(\theta),1}, \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] - (d_{n-1}^1)^2 \\ &= (d_{n-2}^0)^2 + 2(n-2)(d_{n-2}^1)^2 + \frac{(n-2)(n^2-7n+13)}{n-3} (d_{n-2}^2)^2 - (d_{n-1}^1)^2 \\ &= (n-3)^2 ((d_{n-3}^0)^2 + 2d_{n-3}^0 d_{n-4}^0 + (d_{n-4}^0)^2) \\ &\quad + 2(n-2) ((n-2)^2 (d_{n-3}^0)^2 + 2(n-2)(n-3)d_{n-3}^0 d_{n-4}^0 + (n-3)^2 (d_{n-4}^0)^2) \\ &\quad + \frac{(n-2)(n^2-7n+13)}{n-3} ((n-1)^2 (d_{n-3}^0)^2 + 2(n-1)(n-2)d_{n-3}^0 d_{n-4}^0 + (n-2)^2 (d_{n-4}^0)^2) \\ &\quad - (n^2-3n+1)^2 (d_{n-3}^0)^2 - 2(n-1)(n-3)(n^2-3n+1)d_{n-3}^0 d_{n-4}^0 - (n-1)^2 (n-3)^2 (d_{n-4}^0)^2 \\ &= \frac{1}{n-3} \left[ ((n-3)^3 + 2(n-2)^3(n-3) + (n-1)^2(n-2)(n^2-7n+13) \right. \\ &\quad \left. - (n-3)(n^2-3n+1)^2) (d_{n-3}^0)^2 \right. \\ &\quad \left. + 2((n-3)^3 + 2(n-2)^2(n-3)^2 + (n-1)(n-2)^2(n^2-7n+13) \right. \\ &\quad \left. - (n-1)(n-3)^2(n^2-3n+1)) d_{n-3}^0 d_{n-4}^0 \right. \\ &\quad \left. + ((n-3)^3 + 2(n-2)(n-3)^3 + (n-2)^3(n^2-7n+13) - (n-1)^2(n-3)^3) (d_{n-4}^0)^2 \right] \\ &= \frac{n-2}{n-3} ((n+1)(d_{n-3}^0)^2 + 2(n-1)d_{n-3}^0 d_{n-4}^0 + (n-2)(d_{n-4}^0)^2). \end{aligned}$$

Indeed, the first equality comes from Lemma 9, the second one from (16), and the subsequent ones from calculations. Replacing in (17), we obtain

$$\Delta_1 = -\frac{n-2}{(n-1)^2 (n-3) (d_{n-1}^1)^2} ((n+1)(d_{n-3}^0)^2 + 2(n-1)d_{n-3}^0 d_{n-4}^0 + (n-2)(d_{n-4}^0)^2), \quad (18)$$

which is strictly negative because  $n \geq 4$ , and evaluates to

$$-\frac{2}{9 \cdot 1 \cdot 9} (5 \cdot 0 + 2 \cdot 3 \cdot 0 \cdot 1 + 2 \cdot 1) = -\frac{4}{81}$$

for  $n = 4$ .

We next analyze  $\Delta_2$  in a similar way. We first note that

$$\mathbb{P}[B_{\text{AW}(\theta),2} \mid \bar{\mathbf{R}}_{\text{AW}}^{\leq 2}(\theta), \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] = \mathbb{P}[B_{\text{AW}(\theta),2} \mid \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] = \frac{2(n-2)}{(n-1)^2},$$

and

$$\begin{aligned} & \mathbb{P}[B_{\text{FSD}(\theta),2} \mid \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta), \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ &= \frac{\mathbb{P}[B_{\text{FSD}(\theta),2}, \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]}{\mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]} \\ &= \frac{(n-1)!^2 \mathbb{P}[B_{\text{FSD}(\theta),2}, \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]}{(d_{n-1}^1)^2}, \end{aligned}$$

where most equalities follow analogously to the case of  $\Delta_1$ , but now the probability of  $B_{\text{AW}(\theta),2}$  conditional on moving in the first three rounds is  $2(n-2)/(n-1)^2$ , because this event occurs whenever exactly one player starts the third permutation at the home location of the other. We obtain

$$\begin{aligned} \Delta_2 &= \mathbb{P}[B_{\text{AW}(\theta),2} \mid \bar{\mathbf{R}}_{\text{AW}}^{\leq 2}(\theta), \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] \\ &\quad - \mathbb{P}[B_{\text{FSD}(\theta),2} \mid \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta), \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ &= \frac{2(n-2)(d_{n-1}^1)^2 - (n-1)^2(n-1)!^2 \mathbb{P}[B_{\text{FSD}(\theta),2}, \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]}{(n-1)^2(d_{n-1}^1)^2}. \quad (19) \end{aligned}$$

We now compute the numerator, divided by  $n-2$  for convenience:

$$\begin{aligned} & 2(d_{n-1}^1)^2 - \frac{(n-1)^2(n-1)!^2}{n-2} \mathbb{P}[B_{\text{FSD}(\theta),2}, \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\ &= 2 \left( (d_{n-1}^1)^2 - 2d_{n-2}^0 d_{n-2}^1 - (n-3)(d_{n-2}^1)^2 - 2(n-3)d_{n-2}^1 d_{n-2}^2 - \frac{(n-4)(n^2-6n+10)}{n-3} (d_{n-2}^2)^2 \right) \\ &= 2 \left( (n^2-3n+1)^2 (d_{n-3}^0)^2 + 2(n-1)(n-3)(n^2-3n+1)d_{n-3}^0 d_{n-4}^0 + (n-1)^2(n-3)^2 (d_{n-4}^0)^2 \right. \\ &\quad - 2(n-3)((n-2)(d_{n-3}^0)^2 + (2n-5)d_{n-3}^0 d_{n-4}^0 + (n-3)(d_{n-4}^0)^2) \\ &\quad - (n-3)((n-2)^2 (d_{n-3}^0)^2 + 2(n-2)(n-3)d_{n-3}^0 d_{n-4}^0 + (n-3)^2 (d_{n-4}^0)^2) \\ &\quad - 2(n-3)((n-1)(n-2)(d_{n-3}^0)^2 + (2n^2-8n+7)d_{n-3}^0 d_{n-4}^0 + (n-2)(n-3)(d_{n-4}^0)^2) \\ &\quad \left. - \frac{(n-4)(n^2-6n+10)}{n-3} ((n-1)^2 (d_{n-3}^0)^2 + 2(n-1)(n-2)d_{n-3}^0 d_{n-4}^0 + (n-2)^2 (d_{n-4}^0)^2) \right) \\ &= \frac{2}{n-3} \left( ((n-3)(n^2-3n+1)^2 - 2(n-2)(n-3)^2 - (n-2)^2(n-3)^2 - 2(n-1)(n-2)(n-3)^2 \right. \\ &\quad - (n-1)^2(n-4)(n^2-6n+10)) (d_{n-3}^0)^2 \\ &\quad + (2(n-1)(n-3)^2(n^2-3n+1) - 2(n-3)^2(2n-5) - 2(n-2)(n-3)^3 \\ &\quad \left. - 2(n-3)^2(2n^2-8n+7) - 2(n-1)(n-2)(n-4)(n^2-6n+10)) d_{n-3}^0 d_{n-4}^0 \right) \end{aligned}$$

$$\begin{aligned}
& + ((n-1)^2(n-3)^3 - 2(n-3)^3 - (n-3)^4 - 2(n-2)(n-3)^3 \\
& \quad - (n-2)^2(n-4)(n^2 - 6n + 10))(d_{n-4}^0)^2) \\
& = \frac{2}{n-3} ((n+1)(d_{n-3}^0)^2 + 2(n-1)d_{n-3}^0 d_{n-4}^0 + (n-2)(d_{n-4}^0)^2).
\end{aligned}$$

As before, the first equality comes from Lemma 9, the second one from (16), and the subsequent ones from calculations. Replacing in (19), we obtain

$$\Delta_2 = \frac{2(n-2)}{(n-1)^2(n-3)(d_{n-1}^1)^2} ((n+1)(d_{n-3}^0)^2 + 2(n-1)d_{n-3}^0 d_{n-4}^0 + (n-2)(d_{n-4}^0)^2).$$

Since the expression on the right-hand side is strictly positive and twice the one on the right-hand side of (18), this concludes the proof of the (in)equalities regarding  $\Delta_1$  and  $\Delta_2$ .

We now prove that  $\Delta_3 > 0$  when  $n \geq 5$ . We first note that

$$\begin{aligned}
\mathbb{P}[B_{\text{AW}(\theta),3} \mid \bar{\mathbf{R}}_{\text{AW}}^{\leq 2}(\theta), \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] &= \mathbb{P}[B_{\text{AW}(\theta),3} \mid \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] \\
&= \frac{(n-2)(n-3)}{(n-1)^2},
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{P}[B_{\text{FSD}(\theta),3} \mid \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta), \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\
&= \frac{\mathbb{P}[B_{\text{FSD}(\theta),3}, \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]}{\mathbb{P}[\bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]} \\
&= \frac{(n-1)!^2 \mathbb{P}[B_{\text{FSD}(\theta),3}, \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]}{(d_{n-1}^1)^2},
\end{aligned}$$

where most equalities follow analogously to the previous cases, but now the probability of  $B_{\text{AW}(\theta),3}$  conditional on moving in the first three rounds is  $(n-2)(n-3)/(n-1)^2$ , because this event occurs when both players start the third permutation at two different locations, none of which is the home location of the other. We obtain

$$\begin{aligned}
\Delta_3 &= \mathbb{P}[B_{\text{AW}(\theta),3} \mid \bar{\mathbf{R}}_{\text{AW}}^{\leq 2}(\theta), \chi_{\text{AW}(\theta)}^1 = \chi_{\text{AW}(\theta)}^2 = \mathbf{M}^3] \\
&\quad - \mathbb{P}[B_{\text{FSD}(\theta),3} \mid \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta), \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\
&= \frac{(n-2)(n-3)(d_{n-1}^1)^2 - (n-1)^2(n-1)!^2 \mathbb{P}[B_{\text{FSD}(\theta),3}, \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3]}{(n-1)^2(d_{n-1}^1)^2}.
\end{aligned} \tag{20}$$

We now compute the numerator, divided by  $n-2$  for convenience:

$$\begin{aligned}
& (n-3)(d_{n-1}^1)^2 - \frac{(n-1)^2(n-1)!^2}{n-2} \mathbb{P}[B_{\text{FSD}(\theta),3}, \bar{\mathbf{R}}_{\text{FSD}}^{\leq 2}(\theta) \mid \chi_{\text{FSD}(\theta)}^1 = \chi_{\text{FSD}(\theta)}^2 = \mathbf{M}^3] \\
&= (n-3)(d_{n-1}^1)^2 - \frac{2(n^2 - 7n + 13)}{n-3} d_{n-2}^0 d_{n-2}^2 - 2(n-3)(d_{n-2}^1)^2
\end{aligned}$$

$$\begin{aligned}
& - \frac{4(n-4)(n^2-6n+10)}{n-3} d_{n-2}^1 d_{n-2}^2 - \frac{n^4-14n^3+75n^2-185n+181}{n-3} (d_{n-2}^2)^2 \\
= & (n-3)((n^2-3n+1)^2 (d_{n-3}^0)^2 + 2(n-1)(n-3)(n^2-3n+1) d_{n-3}^0 d_{n-4}^0 \\
& + (n-1)^2 (n-3)^2 (d_{n-4}^0)^2) \\
& - 2(n^2-7n+13)((n-1)(d_{n-3}^0)^2 + (2n-3) d_{n-3}^0 d_{n-4}^0 + (n-2)(d_{n-4}^0)^2) \\
& - 2(n-3)((n-2)^2 (d_{n-3}^0)^2 + 2(n-2)(n-3) d_{n-3}^0 d_{n-4}^0 + (n-3)^2 (d_{n-4}^0)^2) \\
& - \frac{4(n-4)(n^2-6n+10)}{n-3} ((n-1)(n-2)(d_{n-3}^0)^2 + (2n^2-8n+7) d_{n-3}^0 d_{n-4}^0 \\
& + (n-2)(n-3)(d_{n-4}^0)^2) \\
& - \frac{n^4-14n^3+75n^2-185n+181}{n-3} ((n-1)^2 (d_{n-3}^0)^2 + 2(n-1)(n-2) d_{n-3}^0 d_{n-4}^0 \\
& + (n-2)^2 (d_{n-4}^0)^2) \\
= & \left( (n-3)(n^2-3n+1)^2 - 2(n-1)(n^2-7n+13) - 2(n-2)^2 (n-3) \right. \\
& \left. - \frac{4(n-1)(n-2)(n-4)(n^2-6n+10)}{n-3} - \frac{(n-1)^2 (n^4-14n^3+75n^2-185n+181)}{n-3} \right) (d_{n-3}^0)^2 \\
& + \left( 2(n-1)(n-3)^2 (n^2-3n+1) - 2(2n-3)(n^2-7n+13) - 4(n-2)(n-3)^2 \right. \\
& \left. - \frac{4(n-4)(n^2-6n+10)(2n^2-8n+7)}{n-3} \right. \\
& \left. - \frac{2(n-1)(n-2)(n^4-14n^3+75n^2-185n+181)}{n-3} \right) d_{n-3}^0 d_{n-4}^0 \\
& + \left( (n-1)^2 (n-3)^3 - 2(n-2)(n^2-7n+13) - 2(n-3)^3 - 4(n-2)(n-4)(n^2-6n+10) \right. \\
& \left. - \frac{(n-2)^2 (n^4-14n^3+75n^2-185n+181)}{n-3} \right) (d_{n-4}^0)^2 \\
= & \frac{(n^3-4n^2+n-2)(d_{n-3}^0)^2 + 2n(n-1)(n-4) d_{n-3}^0 d_{n-4}^0 + (n^3-6n^2+8n-1)(d_{n-4}^0)^2}{n-3}.
\end{aligned}$$

As in the previous cases, the first equality comes from Lemma 9, the second one from (16), and the subsequent ones from calculations. Replacing in (20), we obtain that  $\Delta_3$  equals

$$\frac{(n-2)((n^3-4n^2+n-2)(d_{n-3}^0)^2 + 2n(n-1)(n-4) d_{n-3}^0 d_{n-4}^0 + (n^3-6n^2+8n-1)(d_{n-4}^0)^2)}{(n-1)^2 (n-3) (d_{n-1}^1)^2}.$$

All coefficients in the numerator and all terms in the denominator are strictly positive when  $n \geq 5$ . This is trivial for all linear terms in parentheses; for  $n^3 - 4n^2 + n - 2$  it follows from the fact that  $n^3 > 4n^2$  when  $n \geq 5$ ; for  $n^3 - 6n^2 + 8n - 1$  it follows from the facts that this expression equals 14 when  $n = 5$  and that  $n^3 \geq 6n^2$  when  $n \geq 6$ . We conclude that  $\Delta_3 > 0$ , which finishes the proof.  $\square$

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